OPERATIONS RESEARCH

Unit 1: Linear Programming

1. INTRODUCTION

1.1 TERMINOLOGY

The British/Europeans refer to "operational research", the Americans to "operations research" - but both are often shortened to just "OR" (which is the term we will use).

Another term which is used for this field is "management science" ("MS"). The Americans sometimes combine the terms OR and MS together and say "OR/MS" or "ORMS".

Yet other terms sometimes used are "industrial engineering"("IE"), "decision science" ("DS"), and “problem solving”.

In recent years there has been a move towards a standardization upon a single term for the field, namely the term "OR".

1.2 THE METHODOLOGY OF OR

When OR is used to solve a problem of an organization, the following seven step procedure should be followed:

Step 1. Formulate the Problem: OR analyst first defines the organization's problem. Defining the problem includes specifying the organization's objectives and the parts of the organization (or system) that must be studied before the problem can be solved.

Step 2. Observe the System: Next, the analyst collects data to estimate the values of parameters that affect the organization's problem. These estimates are used to develop (in Step 3) and evaluate (in Step 4) a mathematical model of the organization's problem.

Step 3. Formulate a Mathematical Model of the Problem: The analyst, then, develops a mathematical model (in other words an idealized representation) of the problem. In this class, we describe many mathematical techniques that can be used to model systems.

Step 4. Verify the Model and Use the Model for Prediction: The analyst now tries to determine if the mathematical model developed in Step 3 is an accurate representation of reality. To determine how well the model fits reality, one determines how valid the model is for the current situation.

Step 5. Select a Suitable Alternative: Given a model and a set of alternatives, the analyst chooses the alternative (if there is one) that best meets the organization's objectives. Sometimes the set of alternatives is subject to certain restrictions and constraints. In many situations, the best alternative may be impossible or too costly to determine.

Step 6. Present the Results and Conclusions of the Study: In this step, the analyst presents the model and the recommendations from Step 5 to the decision making individual or group. In some situations, one might present several alternatives and let the organization choose the decision maker(s) choose the one that best meets her/his/their needs.

After presenting the results of the OR study to the decision maker(s), the analyst may find that s/he does not (or they do not) approve of the recommendations. This may result from incorrect definition of the problem on hand or from failure to involve decision maker(s) from the start of the project. In this case, the analyst should return to Step 1, 2, or 3.

Step 7. Implement and Evaluate Recommendation: If the decision maker(s) has accepted the study, the analyst aids in implementing the recommendations. The system must be constantly monitored (and updated dynamically as the environment changes) to ensure that the recommendations are enabling decision maker(s) to meet her/his/their objectives.

1.3 HISTORY OF OR

OR is a relatively new discipline. Whereas 70 years ago it would have been possible to study mathematics, physics or engineering (for example) at university it would not have been possible to study OR, indeed the term OR did not exist then. It was only really in the late 1930's that operational research began in a systematic fashion, and it started in the UK.

Early in 1936 the British Air Ministry established Bawdsey Research Station, on the east coast, near Felixstowe, Suffolk, as the centre where all pre-war radar experiments for both the Air Force and the Army would be carried out. Experimental radar equipment was brought up to a high state of reliability and ranges of over 100 miles on aircraft were obtained.

It was also in 1936 that Royal Air Force (RAF) Fighter Command, charged specifically with the air defense of Britain, was first created. It lacked however any effective fighter aircraft - no Hurricanes or Spitfires had come into service - and no radar data was yet fed into its very elementary warning and control system.

It had become clear that radar would create a whole new series of problems in fighter direction and control so in late 1936 some experiments started at Biggin Hill in Kent into the effective use of such data. This early work, attempting to integrate radar data with ground based observer data for fighter interception, was the start of OR.

The first of three major pre-war air-defense exercises was carried out in the summer of 1937. The experimental radar station at Bawdsey Research Station was brought into operation and the information derived from it was fed into the general air-defense warning and control system. From the early warning point of view this exercise was encouraging, but the tracking information obtained from radar, after filtering and transmission through the control and display network, was not very satisfactory.

In July 1938 a second major air-defense exercise was carried out. Four additional radar stations had been installed along the coast and it was hoped that Britain now had an aircraft location and control system greatly improved both in coverage and effectiveness. Not so! The exercise revealed, rather, that a new and serious problem had arisen. This was the need to coordinate and correlate the additional, and often conflicting, information received from the additional radar stations. With the out-break of war apparently imminent, it was obvious that something new - drastic if necessary - had to be attempted. Some new approach was needed.

Accordingly, on the termination of the exercise, the Superintendent of Bawdsey Research Station, A.P. Rowe, announced that although the exercise had again demonstrated the technical feasibility of the radar system for detecting aircraft, its operational achievements still fell far short of requirements. He therefore proposed that a crash program of research into the operational - as opposed to the technical - aspects of the system should begin immediately. The term "operational research" [RESEARCH into (military) OPERATIONS] was coined as a suitable description of this new branch of applied science. The first team was selected from amongst the scientists of the radar research group the same day.

In the summer of 1939 Britain held what was to be its last pre-war air defense exercise. It involved some 33,000 men, 1,300 aircraft, 110 antiaircraft guns, 700 searchlights, and 100 barrage balloons. This exercise showed a great improvement in the operation of the air defense warning and control system. The contribution made by the OR teams was so apparent that the Air Officer Commander-in-Chief RAF Fighter Command (Air Chief Marshal Sir Hugh Dowding) requested that, on the outbreak of war, they should be attached to his headquarters at Stanmore.

On May 15th 1940, with German forces advancing rapidly in France, Stanmore Research Section was asked to analyze a French request for ten additional fighter squadrons (12 aircraft a squadron) when losses were running at some three squadrons every two days. They prepared graphs for Winston Churchill (the British Prime Minister of the time), based upon a study of current daily losses and replacement rates, indicating how rapidly such a move would deplete fighter strength.

No aircraft were sent and most of those currently in France were recalled. This is held by some to be the most strategic contribution to the course of the war made by OR (as the aircraft and pilots saved were consequently available for the successful air defense of Britain, the Battle of Britain).

In 1941 an Operational Research Section (ORS) was established in Coastal Command which was to carry out some of the most well-known OR work in World War II.

Although scientists had (plainly) been involved in the hardware side of warfare (designing better planes, bombs, tanks, etc) scientific analysis of the operational use of military resources had never taken place in a systematic fashion before the Second World War. Military personnel, often by no means stupid, were simply not trained to undertake such analysis.

These early OR workers came from many different disciplines, one group consisted of a physicist, two physiologists, two mathematical physicists and a surveyor. What such people brought to their work were "scientifically trained" minds, used to querying assumptions, logic, exploring hypotheses, devising experiments, collecting data, analyzing numbers, etc. Many too were of high intellectual caliber (at least four wartime OR personnel were later to win Nobel prizes when they returned to their peacetime disciplines).

By the end of the war OR was well established in the armed services both in the UK and in the USA.

OR started just before World War II in Britain with the establishment of teams of scientists to study the strategic and tactical problems involved in military operations. The objective was to find the most effective utilization of limited military resources by the use of quantitative techniques.

Following the end of the war OR spread, although it spread in different ways in the UK and USA.

You should be clear that the growth of OR since it began (and especially in the last 30 years) is, to a large extent, the result of the increasing power and widespread availability of computers. Most (though not all) OR involves carrying out a large number of numeric calculations. Without computers this would simply not be possible

1.4WHAT IS OPERATIONS RESEARCH?

Operations

- The activities carried out in an organization.

Research

- The process of observation and testing characterized by the scientific method. Situation, problem statement, model construction, validation, experimentation, candidate solutions.

Model

- An abstract representation of reality. Mathematical, physical, narrative, set of rules in computer program.

Systems Approach

- Include broad implications of decisions for the organization at each stage in analysis. Both quantitative and qualitative factors are considered.

Optimal Solution

- A solution to the model that optimizes (maximizes or minimizes) some measure of merit over all feasible solutions.

Team

- A group of individuals bringing various skills and viewpoints to a problem.

Operations Research Techniques

- A collection of general mathematical models, analytical procedures, and algorithms.

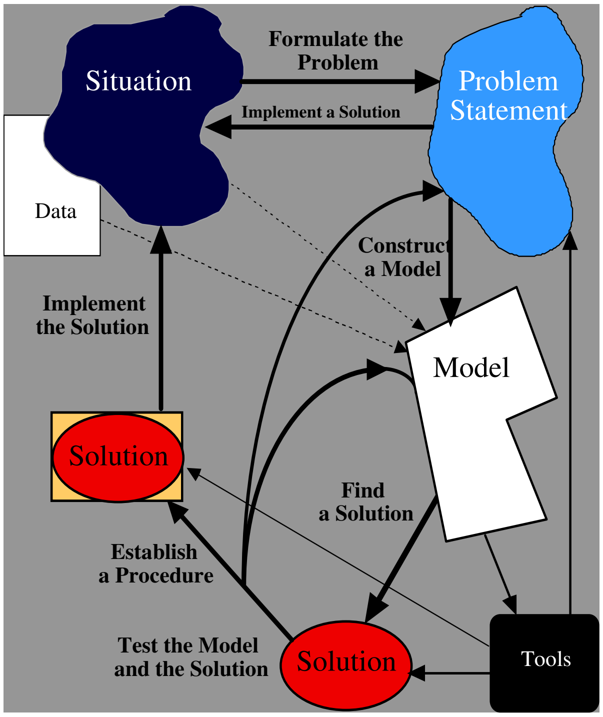
1.5 Definition of OR?

1. OR professionals aim to provide rational bases for decision making by seeking to understand and structure complex situations and to use this understanding to predict system behavior and improve system performance.
2. Much of this work is done using analytical and numerical techniques to develop and manipulate mathematical and computer models of organizational systems composed of people, machines, and procedures.

1.6Problem Solving Process

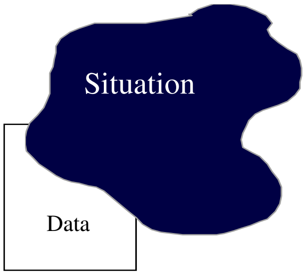
Goal: solve a problem

- Model must be valid
- Model must be tractable
- Solution must be useful



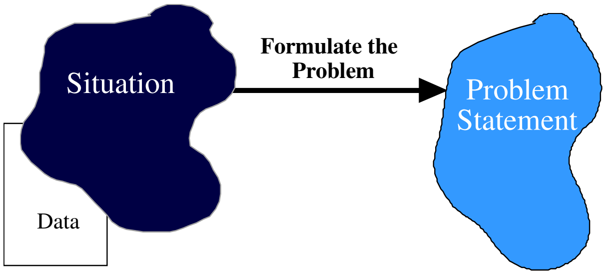
The Situation

- May involve current operations or proposed expansions due to expected market shifts
- May become apparent through consumer complaints or through employee suggestions
- May be a conscious effort to improve efficiency or response to an unexpected crisis.



Example: Internal nursing staff not happy with their schedules; hospital using too many external nurses.

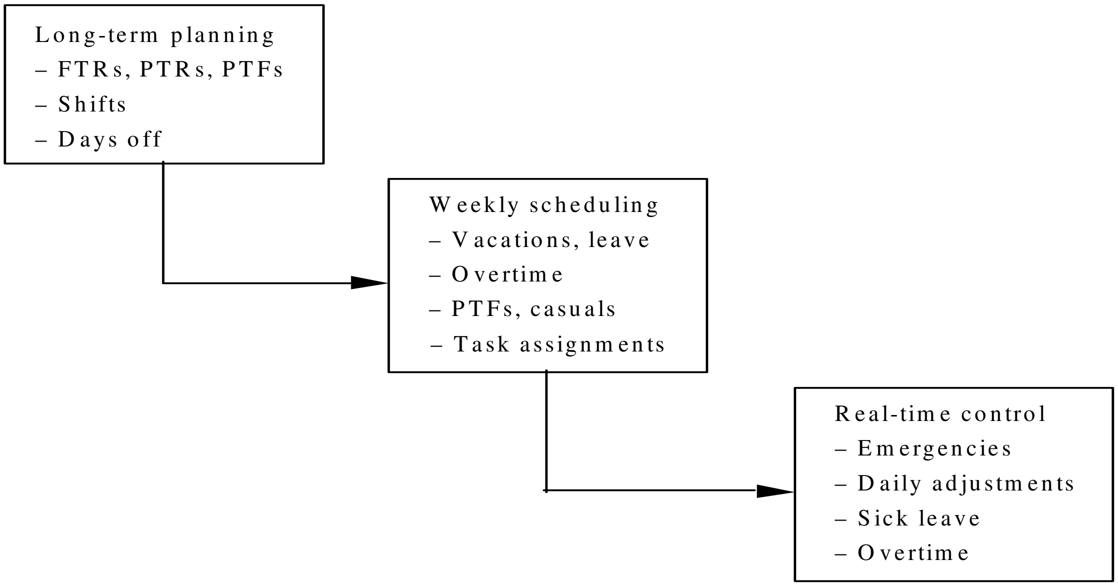
1.6.1 Problem Formulation



- Describe system
- Define boundaries
- State assumptions
- Select performance measures
- Define variables
- Define constraints
- Data requirements

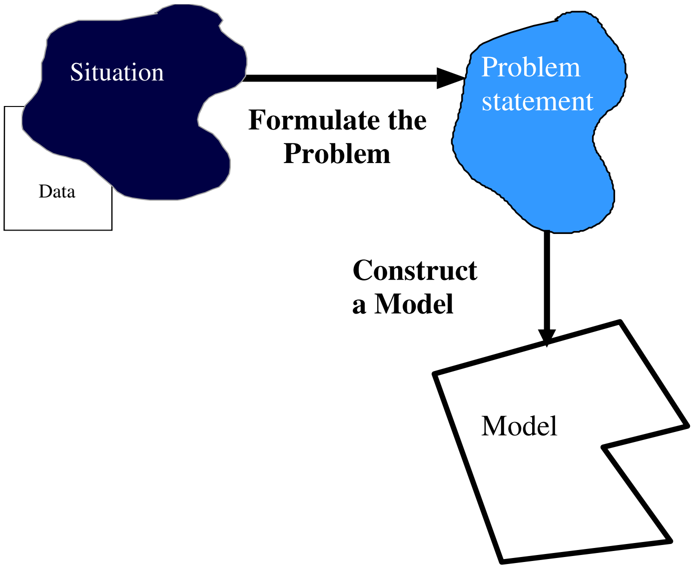
Example: Maximize individual nurse preferences subject to demand requirements.

1.6.2Personnel Planning and Scheduling: Example of Bounding a Problem



1.6.3 Constructing a Model

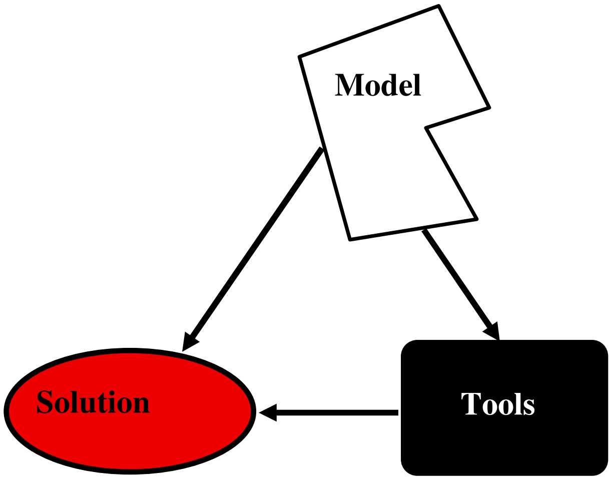
- Problem must be translated from verbal, qualitative terms to logical, quantitative terms
- A logical model is a series of rules, usually embodied in a computer program
- A mathematical model is a collection of functional relationships by which allowable actions are delimited and evaluated.



Example: Define relationships between individual nurse assignments and preference violations; define tradeoffs between the use of internal and external nursing resources.

1.6.5 Solving the Mathematical Model

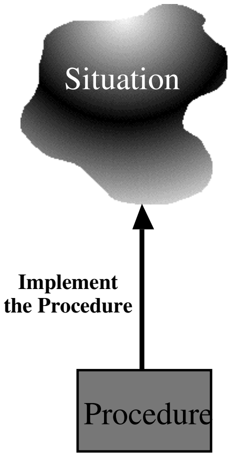
- Many tools are available as will be discussed in this course
- Some lead to “optimal” solutions
- Others only evaluate candidates → trial and error to find “best” course of action



Example: Read nurse profiles and demand requirements, apply algorithm, post-processes results to get monthly schedules.

1.6.6 Implementation

- A solution to a problem usually implies changes for some individuals in the organization
- Often there is resistance to change, making the implementation difficult
- User-friendly system needed
- Those affected should go through training



Example: Implement nurse scheduling system in one unit at a time. Integrate with existing HR and T&A systems. Provide training sessions during the workday.

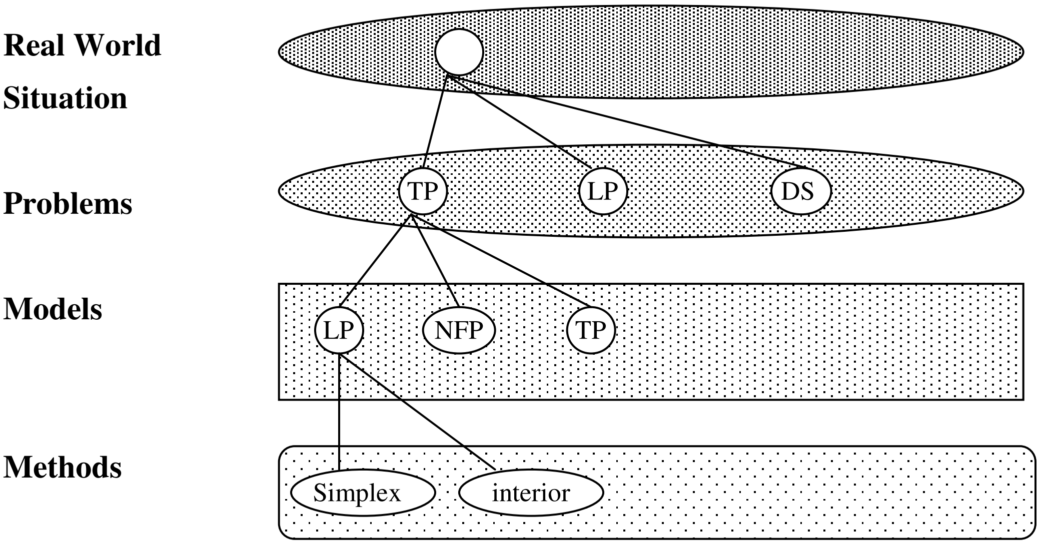
1.7Components of OR-Based Decision Support System:-

Decision support systems based on Operations research models can be of immense value to the decision makers. The Components of a Typical Decision support system are as follows:-

- Data base (nurse profiles, external resources, rules)
- Graphical User Interface (GUI); web enabled using java or VBA
- Algorithms, pre- and post- processor
- What-if analysis
- Report generators

1.7.1 Problems, Models and Methods:-

Operations research consists of the Modeling the real life situations using standard mathematical approaches as problems. These are known as Models. In order to derive solutions to these models specialized techniques such as simplex algorithm, Interior point algorithm etc have been developed. An appreciation of these Problems, Models and Methods forms the Tool kit of Operations Research. The Course on operations research covers the Knowledge on the models and methods using the OR problem solving methodology.



1.7.2 Operations Research Models

Deterministic Models	Stochastic Models
• Linear Programming Chains	• Discrete-Time Markov
• Network Optimization	• Continuous-Time Markov Chains
• Integer Programming	• Queueing
• Nonlinear Programming	• Decision Analysis

1.7.3 Deterministic vs. Stochastic Models

Deterministic models - assume all data are known with certainty

Stochastic models - explicitly represent uncertain data via Random variables or stochastic processes.

Deterministic models involve optimization

Stochastic models - characterize / estimate system performance.

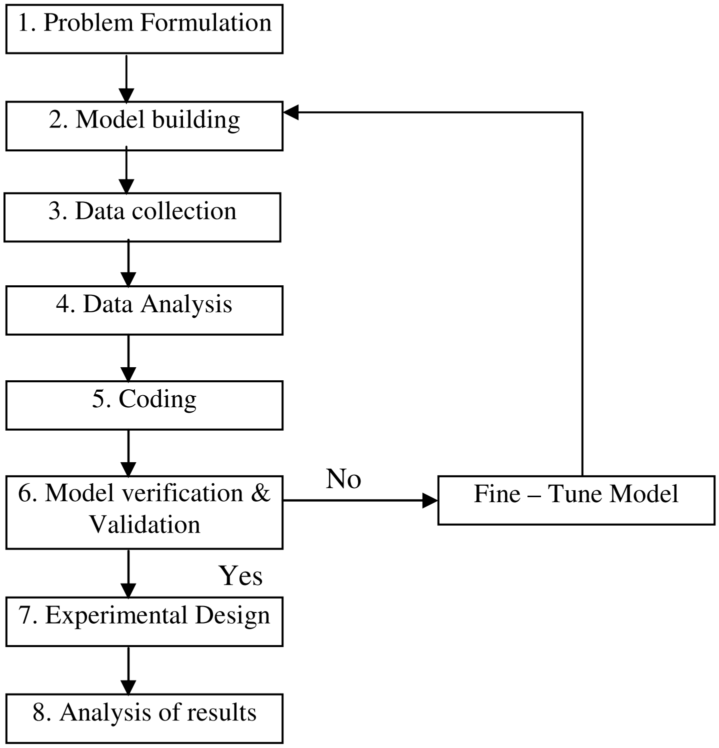
1.7.5 Examples of OR Applications:-

Some typical applications that can be developed using OR methodology includes:-

- Rescheduling aircraft in response to groundings and delays
- Planning production for printed circuit board assembly
- Scheduling equipment operators in mail processing & distribution centers
- Developing routes for propane delivery
- Adjusting nurse schedules in light of daily fluctuations in demand

1.8 Steps in OR Study:-

The typical flow of the OR problem solving methodology is depicted in the flow chart as below.



1.9. BASIC OR CONCEPTS

"OR is the representation of real-world systems by mathematical models together with the use of quantitative methods (algorithms) for solving such models, with a view to optimizing."

We can also define a mathematical model as consisting of:

- Decision variables, which are the unknowns to be determined by the solution to the model.
- Constraints to represent the physical limitations of the system
- An objective function
- An optimal solution to the model is the identification of a set of variable values which are feasible (satisfy all the constraints) and which lead to the optimal value of the objective function.

In general terms we can regard OR as being the application of scientific methods / thinking to decision making.

Underlying OR is the philosophy that:

- decisions have to be made; and
- Using a quantitative (explicit, articulated) approach will lead to better decisions than using non-quantitative (implicit, unarticulated) approaches.

Indeed it can be argued that although OR is imperfect it offers the best available approach to making a particular decision in many instances (which is not to say that using OR will produce the right decision).

Two Mines Example

The Two Mines Company own two different mines that produce an ore which, after being crushed, is graded into three classes: high, medium and low-grade. The company has contracted to provide a smelting plant with 12 tons of high-grade, 8 tons of medium-grade and 24 tons of low-grade ore per week. The two mines have different operating characteristics as detailed below.

Mine	**Cost per day (£'000)**	**Production (tons / day)**		
		High	**Medium**	**Low**
X	180	6	3	4
Y	160	1	1	6

How many days per week should each mine be operated to fulfill the smelting plant contract?

Guessing

To explore the Two Mines problem further we might simply guess (i.e. use our judgment) how many days per week to work and see how they turn out.

- **work one day a week on X, one day a week on Y**

This does not seem like a good guess as it results in only 7 tones a day of high-grade, insufficient to meet the contract requirement for 12 tones of high-grade a day. We say that such a solution is infeasible.

- **work 4 days a week on X, 3 days a week on Y**

This seems like a better guess as it results in sufficient ore to meet the contract. We say that such a solution is feasible. However it is quite expensive (costly).

We would like a solution which supplies what is necessary under the contract at minimum cost. Logically such a minimum cost solution to this decision problem must exist. However even if we keep guessing we can never be sure whether we have found this minimum cost solution or not. Fortunately our structured approach will enable us to find the minimum cost solution.

Solution

What we have is a verbal description of the Two Mines problem. What we need to do is to translate that verbal description into an equivalent mathematical description.

In dealing with problems of this kind we often do best to consider them in the order:

.. Variables

.. Constraints

.. Objective

This process is often called formulating the problem (or more strictly formulating a mathematical representation of the problem).

Variables

These represent the "decisions that have to be made" or the "unknowns".

Let

x = number of days per week mine X is operated

y = number of days per week mine Y is operated

Note here that x >= 0 and y >= 0.

Constraints

It is best to first put each constraint into words and then express it in a mathematical form.

ore production constraints - balance the amount produced with the quantity required under the smelting plant contract

Ore

High $6x + 1y >= 12$

Medium $3x + 1y >= 8$

Low $4x + 6y >= 24$

days per week constraint - we cannot work more than a certain

maximum number of days a week e.g. for a 5 day week we have

$x <= 5$

$y <= 5$

Inequality constraints

Note we have an inequality here rather than an equality. This implies that we may produce more of some grade of ore than we need. In fact we have the general rule: given a choice between an equality and an inequality choose the inequality

For example - if we choose an equality for the ore production constraints we have the three equations $6x+y=12$, $3x+y=8$ and $4x+6y=24$ and there are no values of x and y which satisfy all three equations (the problem is therefore said to be "over-constrained"). For example the values of x and y which satisfy $6x+y=12$ and $3x+y=8$ are $x=4/3$ and $y=4$, but these values do not satisfy $4x+6y=24$.

The reason for this general rule is that choosing an inequality rather than an equality gives us more flexibility in optimizing (maximizing or minimizing) the objective(deciding values for the decision variables that optimize the objective).

Implicit constraints

Constraints such as days per week constraint are often called implicit constraints because they are implicit in the definition of the variables.

Objective

Again in words our objective is (presumably) to minimize cost which is given by 180x + 160y

Hence we have the complete mathematical representation of the problem:

$$
\begin{array}{ll}
\text{minimize} & 180x + 160y \\
\text{subject to} & 6x + y >= 12 \\
 & 3x + y >= 8 \\
 & 4x + 6y >= 24 \\
 & x <= 5 \\
 & y <= 5 \\
 & x,y >= 0
\end{array}
$$

Some notes

The mathematical problem given above has the form

- all variables continuous (i.e. can take fractional values)
- a single objective (maximize or minimize)
- the objective and constraints are linear i.e. any term is either a constant or a constant multiplied by an unknown (e.g. 24, 4x, 6y are linear terms but xy is a non-linear term)

Any formulation which satisfies these three conditions is called a linear program (LP). We have (implicitly) assumed that it is permissible to work in fractions of days - problems where this is not permissible and variables must take integer values will be dealt with under Integer Programming (IP).

Discussion

This problem was a decision problem.

We have taken a real-world situation and constructed an equivalent mathematical representation - such a representation is often called a mathematical model of the real-world situation (and the process by which the model is obtained is called formulating the model).

Just to confuse things the mathematical model of the problem is sometimes called the formulation of the problem.

Having obtained our mathematical model we (hopefully) have some quantitative method which will enable us to numerically solve the model (i.e. obtain a numerical solution) - such a quantitative method is often called an algorithm for solving the model.

Essentially an algorithm (for a particular model) is a set of instructions which, when followed in a step-by-step fashion, will produce a numerical solution to that model.

Our model has an objective that is something which we are trying to optimize. Having obtained the numerical solution of our model we have to translate that solution back into the real-world situation.

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06IP/ IM74 OPERATIONS RESEARCH

Part –A, Unit 1: Linear Programming

(By **Dr.G.Rajendra**, Prof & HOD (IEM), Dr. AIT, Bangalore 560 056)

Definition:

Linear programming (LP) or Linear Programming Problem (LPP)

The general LPP calls for optimizing (MAX/MIN) a linear function of variables called the OBJECTIVE FUNCTION subject to a set of linear equations and or inequalities called for CONSTRAINTS or RESTRICTIONS.

Linear programming problem arises whenever two or more candidates or activities are competing for limited resources.

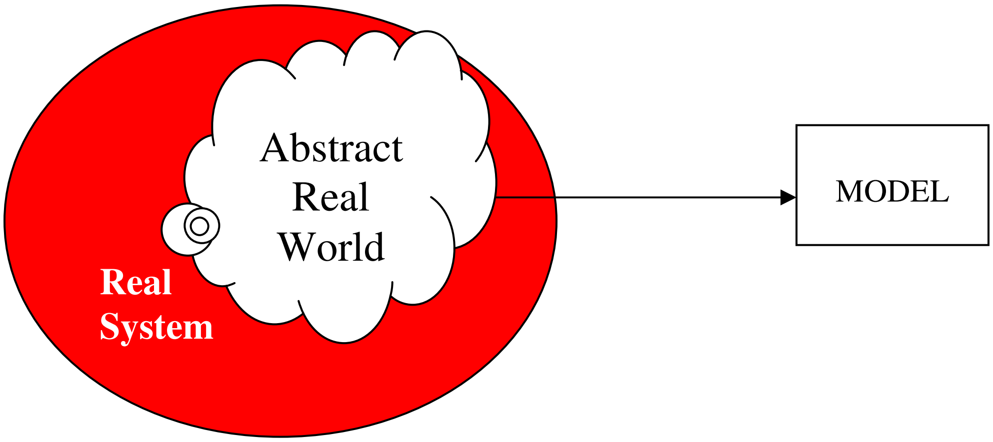
Linear programming applies to optimization technique in which the objective and constraints functions are strictly linear.

Application of Linear Programming
Agriculture, industry, transportation, economics, health Systems, behavioral and social sciences and the military
It can be computerized for 10000 of constraints and variables.

Art of Modeling:

Models are developed as exact representation of real situations in the sense that no approximations are used.

The figure below depicts the level of abstraction from the real situation by concentrating on the dominant variables that control the behavior of the real system.



Levels of abstraction in model development
Example: A Plastic Manufacture company
Step 1: An order is issued to the Production Department
Step 2: Acquires the required raw materials or procures from outside
Step 3: Once the product ion batch is completed the sales department takes Charge of distributing the product to customers.
The overall system, a number of variables involved are

1. Production department: Production capacity expressed in terms of available machine and labor hours in-process inventory and quality control standards.
2. Materials Department: Available stock of raw materials, delivery schedules from outside sources and storage limitations.
3. Sales Department: sales forecast capacity of distribution facilities, effectiveness of the advertisement campaign and effect of competition.

A first level defining the boundaries of the assumed real world and the two dominate variables

1. Production rate.
2. Consumption rate.

Mathematical formulation:

A mathematical program is an optimization problem in which the objective and constraints are given as mathematical functions and functional relationships.

The procedure for mathematical formulation of a LPP consists of the following steps

Step1: write down the decision variables (Products) of the problem

Step2: formulate the objective function to be optimized (maximized or minimized) as linear function of the decision variables

Step3: formulate the other conditions of the problem such as resource limitation, market, constraints, and interrelations between variables etc., linear in equations or equations in terms of the decision variables.

Step4: add non-negativity constraints

The objective function set of constraint and the non-negative constraint together form a Linear Programming Problem.

The general formulations of the LPP can be stated as follows:
In order to find the values of n decision variables
x_1, x_2, x_3---------- x_n to MAX or MIN the objective function.

$$\text{Max } Z = c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n \quad \text{(a)} \longrightarrow \text{Objective Function}$$

Also satisfy m – constraints or Subject to Constraint

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2 \\ |\ | \\ |\ | \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = b_n \\ x_1 >= 0,\ x_2 >= 0,\ \ldots\ldots + x_n >= 0 \end{array}\right\} \begin{array}{l} \text{(b)} \longrightarrow \text{Constant} \\ \\ \text{(c)} \longrightarrow \text{Non Negative Restriction} \end{array}$$

c_j (j = 1,2----n) is the objective function in equations (a) are called cost coefficient (max profit or min cost)
b_i (i= 1,2,----m) defining the constraint requirements or available in equation(b) or available in equation (b) is called stipulations and the constants a_{ij} (i=1,2,----m; j= 1,2,---n)are called structural co-efficient in equation (c) are known as non-negative restriction

Matrix form

$$\underset{(m\times n)}{A =} \left\{\begin{array}{llll} a_{11} & a_{12} & \ldots\ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots\ldots & a_{2n} \\ & & & \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} \end{array}\right\} \qquad \underset{(m\times n)}{X =} \begin{pmatrix} x_1 \\ x_2 \\ -- \\ -- \\ x_n \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ -- \\ -- \\ b_n \end{pmatrix}$$

and $C = (c_{1n}, c_{2n} \ldots\ldots\ldots c_n)$

A is called the coefficient matrix X is the decisions Vector
b is the requirement Vector and c is the profit (cost) vector of the linear program.

The LPP can be expressed in the matrix as follows
Max or Min Z= CX $\longrightarrow$ **Objective Function**
Subject to Constraint
AX=b $\longrightarrow$ **Structural coefficient**
X>=0 $\longrightarrow$ **Non negativity**

Problem 1

A Manufacture produces two types of models M_1 and M_2 each model of the type M_1 requires 4 hrs of grinding and 2 hours of polishing, where as each model of the type M_2 requires 2 hours of grinding and 5 hours of polishing. The manufactures have 2 grinders and 3 polishers. Each grinder works 40 hours a week and each polishers works for 60 hours a week. Profit on M_1 model is Rs. 3.00 and on Model M_2 is Rs 4.00. Whatever produced in week is sold in the market. How should the manufacturer allocate is production capacity to the two types models, so that he may make max in profit in week?

Solutions:

Decision variables. Let X and X be the numbers of units of M_1 and M_2 Model

Objective function: since, the profit on M_1 and M_2 is Rs. 3.0 and Rs 4.

Max $Z = 3x_1 + 4x_2$

Constraint: there are two constraints one for grinding and other is polishing

No of grinders are 2 and the hours available in grinding machine is 40 hrs per week, therefore, total no of hours available of grinders is 2 X 40 = 80 hours

No of polishers are 3 and the hours available in polishing machine is 60 hrs per week, therefore, total no of hours available of polishers is 3 X 60 = 180 hours

The grinding constraint is given by:

$4x_1 + 2x_2 <= 80$

The Polishing Constraint is given by:

$2x_1 + 5x_2 <= 180$

Non negativity restrictions are

$x_1 , x_2 >= 0$ if the company is not manufacturing any products

The LPP of the given problem

Max $Z = 3x_1 + 4x_2$
STC
$4x_1 + 2x_2 <= 80$
$2x_1 + 5x_2 <= 180$
$x_1 , x_2 >= 0$

Problem 2:

Egg contains 6 units of vitamin A per gram and 7 units of vitamin B per gram and cost 12 paise per gram. Milk contains 8 units of vitamin A per gram and 12 units of vitamin B per gram and costs 20 paise per gram. The daily requirements of vitamin A and vitamin B are 100 units and 120 units respectively. Find the optimal product mix.

	EGG	MILK	Min Requirements
Vitamin A	6	8	100
Vitamin B	7	12	120
Cost	12	20	

The LPP of the given Problem

Min $Z = 12x_1 + 20x_2$
STC
$6x_1 + 8x_2 \quad >= 100$
$7x_1 + 12x_2 \; >= 120$
$x_1 , x_2 \; >= 0$

Problem 3: A farmer has 100 acre. He can sell all tomatoes. Lettuce or radishes he raise the price. The price he can obtain is Re 1 per kg of tomatoes, Rs 0.75 a head for lettuce and Rs 2 per kg of radishes. The average yield per acre is 2000kg tomatoes, 3000 heads of lettuce and 1000 kgs of radishes. Fertilizer is available at Rs 0.5 per kg and the amount required per acre 100 kgs each for tomatoes and lettuce, and 50 kgs for radishes. Labor required for sowing, cultivating and harvesting per acre is 5 man-days for tomatoes and radishes, 6 man-days for lettuce. A total of 400 man days of labor available at Rs 20 per man day formulate the problem as linear programming problem model to maximize the farmers' total profit.

Formulation:

Farmer's problem is to decide how much area should be allotted to each type of crop. He wants to grow to maximize his total profit.
Let the farmer decide to allot $X_{1,}$ X_2 and X_3 acre of his land to grow tomatoes, lettuce and radishes respectively.
So the farmer will produce 2000 X_1 kgs of tomatoes, 3000 X_2head of lettuce and 1000 X_3 kgs of radishes.
Profit= sales – cost
= sales – (Labor cost +fertilizer cost)
Sales = $1x\ 2000\,X_1 + 0.75 \text{ x } 3000\,X_2 + 2 \text{ x } 1000\,X_3$
Labor cost = $5x\ 20\,X_1 + 6 \text{ x } 20\,X_2 + 5 \text{ x } 20\,X_3$
Fertilizer cost = $100x0.5\,X_1 + 0.5x\ 100\,X_2 + 0.5x50\,X_3$

Max Z= $1850 X_1 + 2080 X_2 + 1875 X_3$

STC

$X_1 + X_2 + X_3 <= 100$

$5X_1 + 6X_2 + 5X_3 <= 400$

$X_1 , X_2 , X_3 >= 0$

Problem 4:

A Manufacturer of biscuits is considering 4 types of gift packs containing 3 types of biscuits, orange cream (oc), chocolate cream (cc) and wafer's(w) market research study conducted recently to assess the preferences of the consumers shows the following types of assortments to be in good demand.

Assortments	Contents	Selling Price per kg in Rs
A	Not less than 40% of OC Not more than 20% of CC Any quantity of W	29
B	Not less than 20% of OC Not more than 40% of CC Any quantity of W	25
C	Not less than 50% of OC Not more than 10% of CC Any quantity of W	22
D	No restrictions	12

For the biscuits the manufacture capacity and costs are for given below.

Biscuits variety	Plant Capacity Kg/ day	Manufacture cost Rs / Kg
OC	200	8
CC	200	9
W	150	7

Formulate a LP model to find the production schedule which maximizes the profit assuming that there are no market restrictions.

Formulation: the company manufacturer 4 gift packs which oc, cc and w. the quantity of ingredients in each pack is not known.

Let x_{11} denotes the quantities OC of gift pack A
x_{12} denotes the quantities CC of gift pack A
x_{13} denotes the quantities W of gift pack A
x_{21} denotes the quantities OC of gift pack B
x_{22} denotes the quantities CC of gift pack B
x_{23} denotes the quantities W of gift pack B
x_{31} denotes the quantities OC of gift pack C
x_{32} denotes the quantities CC of gift pack C
x_{33} denotes the quantities W of gift pack C
x_{41} denotes the quantities OC of gift pack D
x_{42} denotes the quantities CC of gift pack D
x_{43} denotes the quantities W of gift pack D

The objective Function is to max total profit

Max Z = $20(x_{11} + x_{12} + x_{13}) + 25(x_{21} + x_{22} + x_{23}) + 22(x_{31} + x_{32} + x_{33})$
$+ 12(x_{41} + x_{42} + x_{43}) - 8(x_{11} + x_{21} + x_{31} + x_{41}) - 9(x_{12} + x_{22} + x_{32} + x_{42}) - 7(x_{13}$
$+ x_{23} + x_{33} + x_{43})$

STC
Gift pack A
$x_{11} >= 0.4\,(x_{11} + x_{12} + x_{13})$
$x_{12} <= 0.2\,(x_{11} + x_{12} + x_{13})$
Gift pack B
$x_{21} >= 0.2\,(x_{21} + x_{22} + x_{23})$
$x_{12} <= 0.2\,(x_{21} + x_{22} + x_{23})$
Gift pack C
$x_{31} >= 0.2\,(x_{31} + x_{32} + x_{33})$
$x_{32} <= 0.2\,(x_{31} + x_{32} + x_{33})$

$\sum X_{ij} \sum X_{ij} \sum X_{ij} \sum X_{ij} \geq 0$

Graphical Method:

The graphical procedure includes two steps

1. Determination of the solution space that defines all feasible solutions of the model.
2. Determination of the optimum solution from among all the feasible points in the solution space.

There are two methods in the solutions for graphical method

- Extreme point method
- Objective function line method

Steps involved in graphical method are as follows:

- Consider each inequality constraint as equation
- Plot each equation on the graph as each will geometrically represent a straight line.
- Mark the region. If the constraint is <= type then region below line lying in the first quadrant (due to non negativity variables) is shaded.
  If the constraint is >= type then region above line lying in the first quadrant is shaded.
- Assign an arbitrary value say zero for the objective function.
- Draw the straight line to represent the objective function with the arbitrary value
- Stretch the objective function line till the extreme points of the feasible region. In the maximization case this line will stop farthest from the origin and passing through at least one corner of the feasible region.

In the minimization case, this line will stop nearest to the origin and passing through at least one corner of the feasible region.

- Find the co-ordination of the extreme points selected in step 6 and find the maximum or minimum value of Z.

Problem 1

Max Z $=3x_1 + 5x_2$

STC

$x_1 <= 4$

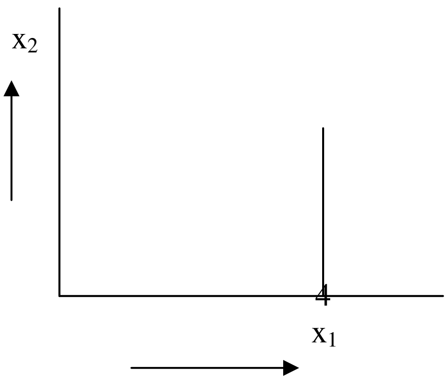
$2x_2 <=12$

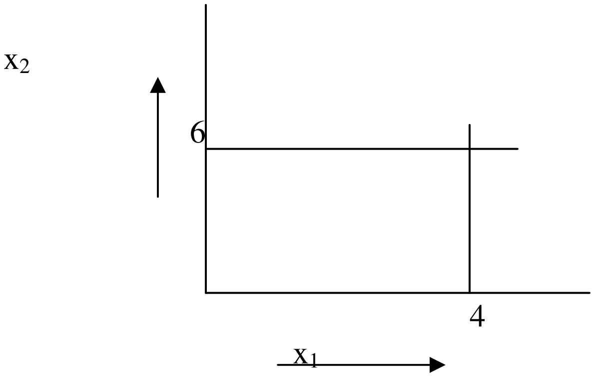
$3x_1 + 2x_2 <= 18$

$x_1, x_2 >= 0$

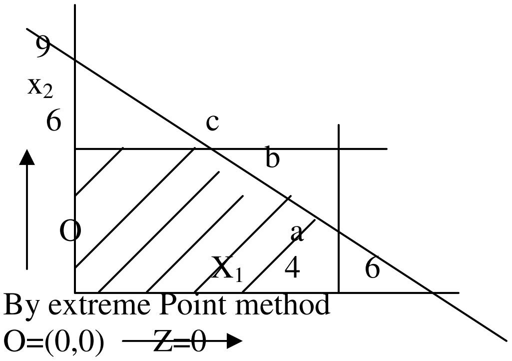
Solution

$x_1 <= 4 \Rightarrow x_1 = 4$ $\quad$ $2x_2 <=12$ $\quad$ $2x_2=12 \Rightarrow x_2 =6$



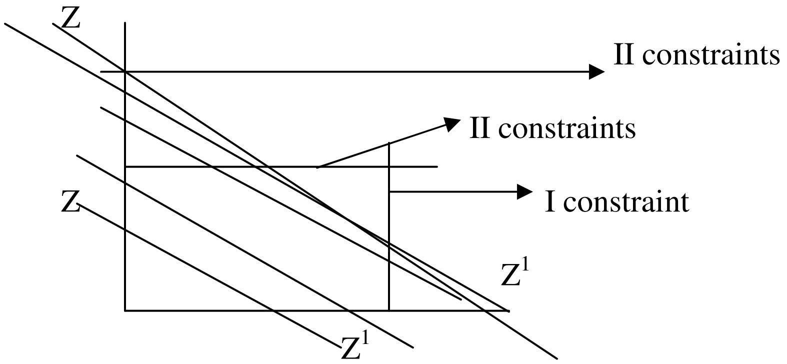


$3x_1 + 2x_2 <= 18 \Rightarrow$ put $x_1 =0$, $x_1 = 6$, put $x_2 = 0$, $x_2 = 9$



By extreme Point method

O=(0,0)	Z=0
A=(4,0)	Z=12
B=(4,3)	Z=12+15=27
C=(2,6)	Z= 6+30 =36
D= (0,6)	Z= 30



By objective function line method,

To find the point of Max value of Z in the feasible region we use objective function line as ZZ^1 , same type of lines are used for different assumed Z value to find the Max Z in the solution space as shown in the above figure.

Let us start = 10
Max Z=10 =$3x_1 + 5x_2$ this will show the value as (3.33,2) by plotting this points in the solution space it explains that Z must be large as 10 we can see many points above this line and within the region.

When Z=20
Max Z=20 =$3x_1 + 5x_2$ this will show the value as (6.66, 4) by plotting this points in the solution space z must be at least 20. The trial and error procedure involves nothing more than drawing a family of parallel lines containing at least one point in the permissible region and selecting the distance from the origin. This lines passes through the points (2,6) or Z=36

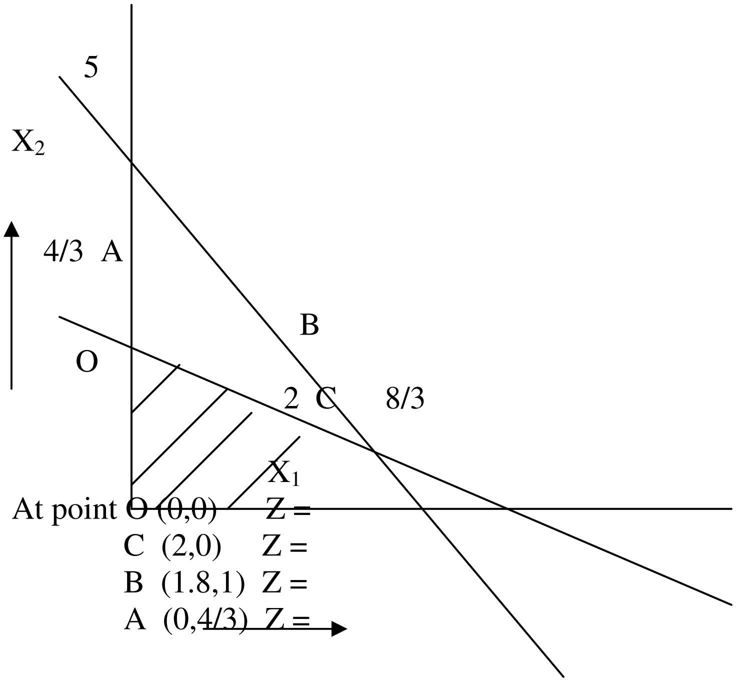
Max Z = 36= $3x_1 + 5x_2$ the points are (12, 7.2) this points lies at the intersection of the 2 lines $2x_2$ =12 and $3x_1 + 2x_2$ = 18. so, this point can be calculated algebraically as the simultaneous solutions of these 2 equation.
Conclusions:

The solution indicates that the company should produce products 1 & 2 at the rate of 2 per minute an d6 / minute respectively with resulting profitable of 36 / minute.

No other mix of 2 products would be profitable according to the model.

Problem 1: find the max Value of the given LPP

Max Z =$x_1 + 3x_2$
STC
$3x_1 + 6x_2$ <= 100 ⟹ (8/3, 4/3)
$5x_1 + 2x_2$ <= 120 ⟹ (2,5)
x_1 , x_2 >= 0



At point O (0,0) Z =
C (2,0) Z =
B (1.8,1) Z =
A (0,4/3) Z =

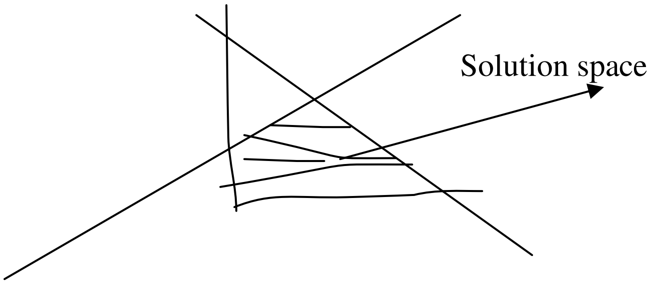
Problem 2: find the max Value of the given LPP
Max Z =5x1 + 2x2
STC
x 1 + x2 <= 4
3x1 + 8x2 <= 24
10x1 +78x2 <= 35
x1 , x2 >= 0

Problem 3: find the max Value of the given LPP
Max Z = - x1 + 2x2
STC
-x1 +3x2 <= 10
x1 +x2 <= 6
x1 - x2 <= 2
x1 , x2 >= 0

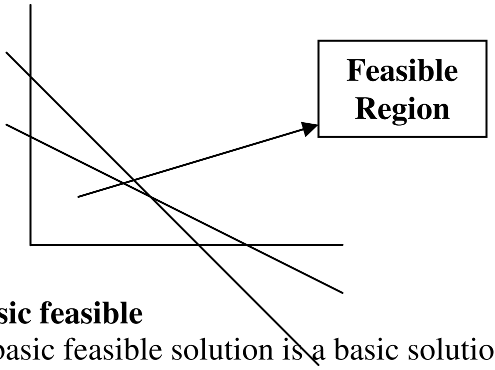
Problem 4: find the max Value of the given LPP
Max Z =7x1 + 3x2
STC
x1 + 2x2 <= 3
x1 + x2 <= 4
0<= x1 <=5/2
0 <= x2 <= 3/2
x1 , x2 >= 0

Problem 5: find the max Value of the given LPP
Max Z =20x1 + 10x2
STC
x1 +2x2 <= 40
3x1 + x2 <= 30
4x1 + 3x2 <= 30
x1 , x2 >= 0

Solution space
Solutions mean the final answer to a problem
Solutions space to a LPP is the space containing such points. The co-ordinates of which satisfy all the constraints simultaneously. The region of feasibility of all the constraints including non-negativity requirements.



Feasible:
The feasible region for an LP is the set of all points that satisfies all the LPs constraints and sign restrictions.



Basic feasible
A basic feasible solution is a basic solution which also satisfies that is all basic variables are non-negative.

Example:
$4x_1 + 2x_2 <= 80$ we add x3 as slack variable
$2x_1 + 5x_2 <= 180$ we add x4 as slack variable

$\begin{bmatrix} 4 & 2 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is unit matrix is called basic feasible solution

Optimal

Any feasible solution which optimizes (Min or Max) the objective function of the LPP is called its optimum solution.

Infeasible / Inconsistency in LPP

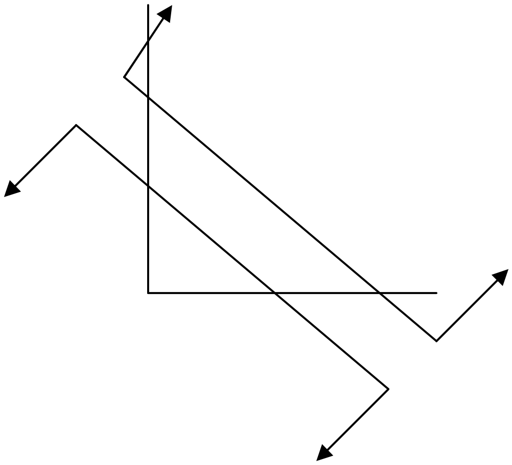
Inconsistency also known as infeasibility

The constraint system is such that one constraint opposes one or more. It is not possible to find one common solutions to satisfy all the constraints in the system.

Ex:-

$2x_1 + x_2 <= 20$ → (10,20)

$2x_1 + x_2 <= 40$ → (20,40)



If both the constraint cannot be satisfied simultaneously. Such constraint system is said to be raise to inconsistency or infeasibility

Redundancy;

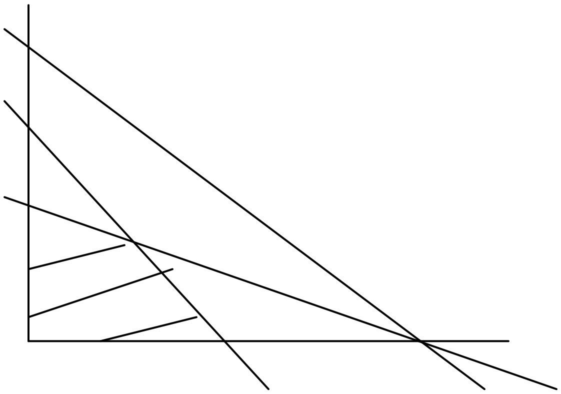
A set of constraint is said to be redundant if one or more of them are automatically satisfied on the basis of the requirement of the others.

Ex:

$2x_1 + x_2 <= 20$ → (10,20)

$2x_1 + x_2 <= 35$ → (17.5, 35)

$x_1 + 2x_2 <= 20$ → (20,10)



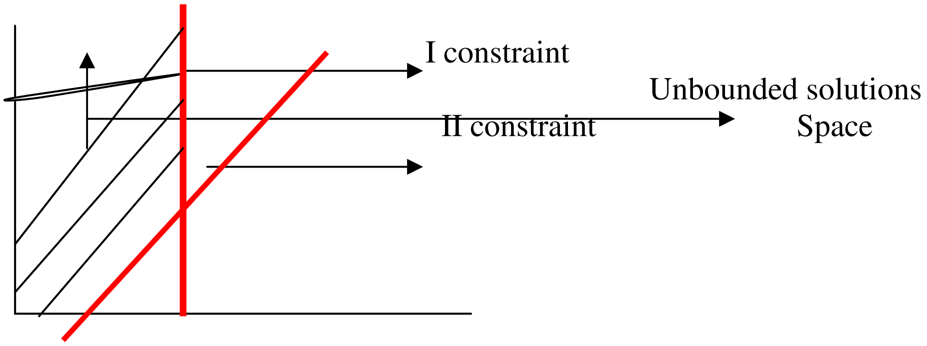
A redundant constraint system is one in which deletion of at least one of the constraint will not alter the solution space.

Degeneracy

A basic feasible solution is said to degenerate if one or more basic variable are zero.
The values of the variables may be increases indefinitely without violating any of the constraint i.e., the solution space is unbounded in at least one direction.
As result the objective function value may increase or decrease indefinitely.
Ex;

$x_1 - x_2 <= 10 \longrightarrow (10,-10)$

$2x_1 <=40 \longrightarrow (20,0)$



Standard form

The standard form of a linear programming problem with m constraints and n variables can be represented as follows:

$$\text{Max } Z = c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n$$

Also satisfy m – constraints or Subject to Constraint

$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2$

| |

| |

$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = b_n$

$x_1>=0 , x_2 >=0, \ldots\ldots + x_n >=0$

The main features of the standard form

1. the objective function is of the maximization or minimization type
2. all constraints are expressed as equations
3. all variables are restricted to be nonnegative
4. The right-hand side constant of each constraint is nonnegative.

Now, considering how a LPP can be formulated in the standard form will be as follows:

Case (a): if a problem aims at minimizing the objective function. Then it can be converted into a maximization problem simply by multiplying the objective by (-1)

Case (b): if a constraint is of <= type , we add a non negative variable called slack variables is added to the LHS of the constraint on the other hand if the constraint is of >= type, we subtract a non-negative variable called the surplus variable from the LHS.

Case (c) when the variables are unrestricted in sign it can be represented as
$X_j = X^1{}_j - X^{11}{}_j$ or $X_1 = X^1{}_1 - X^{11}{}_1$

It may become necessary to introduce a variable that can assume both +ve and –ve values. Generally, unrestricted variable is generally replaced by the difference of 2 non -ve variables.

Problem 1:

Rewrite in standard form the following linear programming problem
$\text{Min } Z = 12x_1 + 5x_2$
STC
$5x_1 + 3x_2 \geq 15$
$7x_1 + 2x_2 \leq 14$
$x_1, x_2 \geq 0$

Solution:

Since, the given problem is minimization then it should be converted to maximization by just multiply by (-1) and the first constraint is >= type it is standard by adding by surplus variable as $x_3 \geq 0$ and 2nd constraint is <= type it is standard by adding slack variable and then the given problem is reformulated as follows:

$\text{Max } Z = -12x_1 - 5x_2 - 0x_3 + 0x_4$
STC
$5x_1 + 3x_2 - x_3 = 15$
$7x_1 + 2x_2 + x_4 = 14$
$x_1, x_2, x_3, x_4 \geq 0$

The matrix form
$\text{Max } Z = (-12, -5, 0, 0)\,(-x_1, -x_2, -x_3, x_4)$
STC

$$\begin{Bmatrix} 5 & 3 & -1 & 0 \\ 7 & 2 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 15 \\ 14 \end{Bmatrix}$$

$x_1, x_2, x_3, x_4 \geq 0$

Problem 2:

Rewrite in standard form the following linear programming problem
Max Z =$2x_1 + 5x_2 + 4x_3$
STC
$-2x_1 + 4x_2 <= 4$
$x_1 + 2x_2+x_3 >=5$
$2x_1 + 3x_3 <= 2$
x_1 , x_2 >= 0 x_3 is unrestricted in sign

Solution:
In the given problem it is maximization problem and the constraint are of in equations. The first constraint is <= type we introduce slack variable as x_4 >=0, 2nd constraint is of >= type, we introduce surplus variable as x_5 >=0 and third constraint is <= type we introduce slack variable as x_6 >=0. the x_3 variable is un restricted in sign. So, this can be written as $X_3 = X^1{}_3 – X^{11}{}_3$
Then, the given LPP is rewritten as
Max Z =$2x_1 + 5x_2 + 4X^1{}_3 – 4X^{11}{}_3 + 0x_4 - 0x_5+ 0x_6$
STC
$-2x_1 + 4x_2 + x_4 = 4$
$x_1 + 2x_2+ X^1{}_3 – X^{11}{}_3 - x_5 =5$
$2x_1 + 3 X^1{}_3 – 3X^{11}{}_3 + 0x_6= 2$
$x_1 , x_2 , x^1{}_{3,}\ x^{11}{}_{3,}\ x_4 ,x_{5,}\ x_6 >= 0$

The matrix form
Max Z = (2,5,4,-4,0,0,0) ($x_1 , x_2 , x^1{}_{3,}\ x^{11}{}_{3,}\ x_4 ,x_{5,}\ x_6$)
STC

$$\left\{\begin{array}{ccccccc} -2 & 4 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & -1 & 0 & -1 & 0 \\ 2 & 0 & 3 & -3 & 0 & 0 & 1 \end{array}\right\} \left\{\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right\} = \left\{\begin{array}{c} 4 \\ 5 \\ 2 \end{array}\right\}$$

$x_1 , x_2 , x^1{}_{3,}\ x^{11}{}_{3,}\ x_4 ,x_{5,}\ x_6 >= 0$

In order to develop a general solutions for LPP having more than 2 variables the LPP must be put in the standard form.

The ideas conveyed by the graphical LP solution lay the foundation for the development of the algebraic simplex method.

Graphical Method	Algebraic simplex method
Solution space consists of infinity of feasible points	The system has infinity of feasible solution
Candidate for the optimum solution are given by a finite number of corner points	Candidate for the optimum solution are given by a finite number of basic feasible solution
Use the objective function to determine the optimum corner point	Use the objective function to determine the optimum basic feasible solution

Principles of the simplex method.

The simplex method is developed by G.B Dantzig is an iterative procedure for solving linear programming problem expressed in standard form. In addition to this simplex method requires constraint equations to be expressed as a canonical system form which a basic feasible solution can be readily obtained

The solutions of any LPP by simplex algorithm the existence of an IBFS is always assumed, the following steps help to reach an optimum solution of an LPP.

Procedure for Maximization

Step 1 write the given LPP in standard form

Step 2 check whether the objective function of the given LPP is to be Maximized or minimized if its to be minimized then we convert it into a problem of maximizing using the result.

Min z = - Max Z or (-Z)

Step 3 check whether all b_i (i=123…..m0 are non-negative. if anyone b_i is –VE, then multiply the corresponding in equation of the constraint by -1.

Step 4 Convert all the in equations of the constraint into equations by introducing slack or surplus variable in the constraints. Put these costs equal to zero in objective cost.

Step 5 Obtain an IBFS to problem in the form identity matrix form in canonical form

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and put it in the 1st column of simples table.

Step 6 compute the net evaluation row $(Z_j - C_j)$ (j=1,2,3….n)

$Z_j - C_j$ = P/U (Profit / unit) x X_j $-C_j$ (j=1,2,3…n)

examine the sign of $Z_j - C_j$

i)if all $(Z_j - C_j) >= 0$ then the IBFS solutions column is an optimum basic feasible solutions.

ii) if at least one $(Z_j - C_j) <=0$ proceed to the next step

Step 7 if there are more than one –ve $(Z_j - C_j)$ then choose the most –ve of them then it will become key column

i)if all the no's in the key column is –ve then there is an unbounded solutions to the given problem

ii)if at least one $X_m >0$ (m =123…n) then the corresponding vector $X_m >=0$ (m=1,2,3…n) then the corresponding vector X_m entry the basis of solution column

Step 8 compute the ratios = solutions column no / key column no.
And choose the minimum of them. Let the minimum of these ratios be the key row these variable in the basic variable column of the key row will become the leaving element or variable.

Step 9 using the below relation to find new no of other than key row and new no for key row also

New no for pivot row = current pivot row / pivot element
Other than key row
New element = old element – (PCE * NPRE)
PCE= Pivot column element, NPRE=new pivot row element
New no= old no – (corresponding Key column / Key element)
x (corresponding key row)

Step 10 go to step5 and repeat the computational procedure until either an optimum solutions is obtained or there is an indication of an unbounded solution.

Note : case 1 in case of a tie for entering basis vector. i.e., there are 2 or more $Z_j - C_j$ which are equal and at the same time the highest –ve values then arbitrary selection of any one of them will not alter optimality.

Case 2 in case of a tie for the leaving variable i.e., there are 2 or more min ratio column i.e., (solution no / key column no) which are equal and greater than zero then arbitrary select any one of them will not alter optimality. But, if the tied ratios are zeros then charnes method of penalty should be followed.

Problem 1:

Use simplex method to solve the given LPP

Max Z =5x_1 + 3x_2

STC

$x_1 + x_2 <= 2$

$5x_1 + 2x_2 <= 10$

$3x_1 + 8x_2 <= 12$

$x_1, x_2 >= 0$

Solution:

Step 1: Since the problem is maximization problem all the constraint are <= type and the requirements are +ve. This satisfies the simplex method procedure.

Step 2: since all the constraints are <= type we introduce the slack variables for all the constraints as $x_3 >=0$, $x_4 >=0$, $x_5 >=0$ for the I II and III constraint

Step 3: the given LPP can be put in standard form

Max Z =5x_1 + 3x_2 + (0) x_3+ (0) x_4 + (0)x_5

STC

$x_1 + x_2 + x_3 <= 2$

$5x_1 + 2x_2 + x_4 <= 10$

$3x_1 + 8x_2 + x_5 <= 12$

$x_1, x_2, x_3, x_4, x_5 >= 0$

Step 4: matrix form

Max Z = (5,3,0,0,0) $(x_1, x_2, x_3, x_4, x_5)$

STC

$$\begin{Bmatrix} 1 & 1 & 1 & 0 & 0 \\ 5 & 2 & 0 & 1 & 0 \\ 3 & 8 & 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 10 \\ 12 \end{Bmatrix}$$

$x_1, x_2, x_3, x_4, x_5 >= 0$

Step 5 : since, considering sub-matrix from the matrix are which form basic variables for the starting table of simplex
(1 0 0) (0 1 0) (0 0 1) are linearly independent column vectors of A.
Therefore, the sub Matrix is

$$B= \left\{ \begin{matrix} 1\ 0\ 0 \\ 0\ 1\ 0 \\ 0\ 0\ 1 \end{matrix} \right\}$$

The corresponding variables of the sub matrix is (x_3, x_4 ,x_5) and these variables are the basic variables for the starting iteration of the simplex problem and there obvious initial basic feasible solutions are (x_3, x_4 ,x_5) = (2, 10, 12)

			5	3	0	0	0	
Basic Variable (V)	Profit / Unit (P/U)	solution	x_1	x_2	x_3	x_4	x_5	Min Ratio = soln. no /key column No.
	0	2	1	1	1	0	0	2/1 = 2
x_3	0	10	5	2	0	1	0	10/5 = 2
x_5	0	12	3	8	0	0	1	12/3 = 4
	Max Z= 0x2+0x10 +0x12		=0x1 +0x5 +0x3 -5	=0x1 +0x2 +0x8 -3	=0x1 +0x0 +0x0 -0	=0x0 +0x1 +0x0 -0	=0x0 +0x0 +0x1 -0	
			-5	-3	0	0	0	

Leaving variable

Key Row

Key Column

Entering Variable

5 3 0 0 0

Basic Variable (B V)	Profit / Unit (P/U)	solution	x_1	x_2	x_3	x_4	x_5	Min Ratio = soln. no /key column No.
x_1	5	2	1	1	1	0	0	
x_4	0	0	0	-3	-5	1	0	
x_5	0	6	0	5	-3	0	1	
	Max Z= 5x2+0x0+0x6 =10		=5x1+0x0 +0x0-5	=5x1+0x-3 +0x5 -3	=5x1 +0x-5 + 0x-3 - 0	=5x0+0x1+0x0-0	=5x0+0x0+0x1-0	
			0	2	5	0	0	

New Numbers for Key row
Soln.= old no / Key element
= 2/ =2
x_1 = 1/1 = 1
x_2 = 1/1 = 1
x_3 = 1/1 = 1
x_4 = 0/1 = 0
x_5 = 0/1 = 0

other than key rows new no is found by using the following formulae

New No=old element – PCE*NPRE

for x_4 row new no are
Soln. = 10-5*2= 0
x_1 = 5-5*1 = 0
x_2 = 2-5*1 = -3
x_3 = 0-5*1 = -5
x_4 = 1-5*0 =1
x_5 = 0 -5*0 =0

for x_5 row the new no are
Soln. = 12-3*2 =6
x_1 =3-3*1 = 0
x_2 = 8-3*1 = 5
x_3 = 0-3*1 = -3
x_4 = 0-3*0 = 0
x_5 = 1 -3*0 = 1

Since, the given problems net evaluation row is +ve, then given problem as attained the optimum

Therefore, $x_1 =2$, $x_2 =0$, $x_3 =0$, $x_4 =0$, $x_5 =6$,

Substitute in the objective function

Max $Z =5x_1 + 3x_2 + (0)\, x_3 + (0)\, x_4 + (0)x_5$

Max $Z =5x2+3x0+0x0+0x0+0x6$

Max $Z = 10$

Problem no 2

Solve the given problem by simplex method

Max $Z =107x_1 + x_2 +2\, x_3$

STC

$14x_1 + x_2 - 6x_3 +3x_4 \quad = 7$

$16x_1 + 1/2x_2 -6x_3 \quad <= 5$

$16x_1 - 8x_2 - x_3 \quad <= 0$

$x_1,\ x_2, x_3, x_4 >= 0$

Solutions:

In the given problem the objective function is MaxZ and it has only three variables.

The I constraint is of standard form already slack variable is introduced as $x_4 >= 0$ and the value should be one but, it is having 3 due this it should be divided by three for enter equation on both sides and II & III constraint are of <= type so we introduce $x_5 >= 0$ $x_6 >= 0$ as slack variable.

Then the given problem can be rewritten as

Max $Z =107x_1 + x_2 +2\, x_3 + 0x_4 +0\, x_5 +0\, x_6$

STC

$14/3x_1 + 1/3\, x_2 – 6/3x_3 +3/3x_4 \quad = 7/3$

$16x_1 + 1/2x_2 - 6x_3 + x_5 \quad = 5$

$16x_1 - 8x_2 - x_3 + x_6 \quad = 0$

$x_1, x_2, x_3, x_4\ x_5\ x_6 \quad >= 0$

The matrix form

Max Z= (107, 1, 2, 0, 0, 0) ($x_1, x_2, x_3, x_4, x_5, x_6$)

STC

$$\begin{Bmatrix} 14/3 & 1/3 & -2 & 1 & 0 & 0 \\ 16 & 1/2 & -6 & 0 & 1 & 0 \\ 16 & -1 & -1 & 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix} = \begin{Bmatrix} 7/3 \\ 5 \\ 0 \end{Bmatrix}$$

$x_1, x_2, x_3, x_4\ x_5\ x_6 \ >= 0$

$$\begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix}$$

Now, the canonical form from the above matrix is the corresponding variables are ($x_4\ x_5\ x_6$) and their obvious solution is (7/3 5 0)

	Starting Table:		107	1	2	0	0	0	
Basic variable (B.V)	Profit /unit P/U	Solution	x_1	x_2	x_3	x_4	x_5	x_6	Min ratio =Soln. no/ Pivot column no
x_4	0	7/3	14/3	1	-6	1	0	0	=7/3 / 14/3=0.5
x_5	0	5	16	½	-6	0	1	0	=16/5= 3.2
x_6	0	0	16	-1	-1	0	0	1	=16/0=∞
	Max Z = 0 x 7/3 + 0 x 5 + 0 x 0 = 0		=(0x14/3 +0x16+0x 16) -107	=(0x1+0 x1/2+0x -1) -1	= (0 x-6 +0x- 6+0x-1) -2	=(0x1 +0x0 +0x0) - 0	=(0x0+ 0x1+0x 0)- 0	=(0x0+0 x0+0x1) - 0	
			-107	-1	-2	0	0	0	

First Table 1:

107 1 2 0 0 0

Basic variable (B.V)	Profit /unit P/U	Solution	x_1	x_2	x_3	x_4	x_5	x_6	Min ratio =Soln. no/ Pivot column no
x_1	107	0.5	1	3/14	1	3/14	0	0	
x_5	0	-3	0	-41/14	-22	-48/14	1	0	
x_6	0	-8	0	-62/14	- 17	-48/14	0	1	
	Max Z = 107 x 0.5 + 0 x -3 +0 x -8 = 0		=(1x107 +0x0 +0x0) -107	=(107x3 /14 + 0 x -41/14 + 0 x - 62/14) - 1	= (107 x 1 + 0 x - 22 + 0 x-17) -2	=(107x3 /14 + 0 x -48/14 + 0 x - 48/14) - 0	=(107 x 0 +0 x 1+ 0 x 0) - 0	=(107x0 + 0x0 +0x1) - 0	
	Max Z=53.5		0	+ve	+ve	+ve	+ve	0	

New Numbers for Key row

Soln.= old no / Key element

= 7/3/ 14/3 =0.5

x_1 = 14/3 / 14/3 = 1

x_2 = 1/14/3 = 3/14

x_3 = -6 / 14/3 = 1

x_4 = 1/14/3 = 3/14

x_5 = 0/14/3 = 0

x_6 = 0/14/3 = 0

for x_6 row the new no are

Soln. = 0-16*0.5 = -8

x_1 =16-16*1 = 0

x_2 = -1-16*3/14 = -62/14

x_3 = -1-16*1 = -17

x_4 = 0-16*3/14 = -48/14

x_5 = 0 -16*0 = 0

x_6 = 1 -16*0 = 1

other than key rows new no is found by using the following formulae

New No=old element – PCE*NPRE

for x_5 row new no are

Soln. = 5-16*0.5= -3

x_1 = 16-16*1 = 0

x_2 = 1/2-16*3/14 = -41/14

x_3 = -6-16*1 = -22

x_4 = 0-16*3/14= - 48/14

x_5 = 1 -16*0 =1

x_6 = 0 -16*0 =0

Since, all the NER is positive then given problem is optimal

Therefore, $x_1 = 0.5$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, $x_5 = -3$, $x_6 = -8$

Max Z = 107*0.5 + 1*0 +2*0+ 0*0 +0*-3 +0* -8

= 53.5

Problem 3:

Solve the given LPP by simplex method

Max $Z = 4x_1 + 5x_2 + 9x_3 + 11x_4$

STC

$x_1 + x_2 - 6x_3 + 3x_4 \quad <= 7$

$7x_1 + 5x_2 + 3x_3 + 2x_4 \quad <= 120$

$3x_1 + 5x_2 + 10x_3 + 15x_4 \quad <= 100$

$x_1, x_2, x_3, x_4 >= 0$

Solution:

The given problem is maximization problem and all the constraints are <= type, so, we introduce the slack variable as $x_5 >= 0$,

$x_6 >= 0$, $x_7 >= 0$ for I , II and III constraints respectively.

The given LPP can be written as

Max $Z = 4x_1 + 5x_2 + 9x_3 + 11x_4$

STC

$x_1 + x_2 + x_3 + x_4 + x_5 \quad = 7$

$7x_1 + 5x_2 + 3x_3 + 2x_4 + x_6 \quad = 120$

$3x_1 + 5x_2 + 10x_3 + 15x_4 + x_7 \quad = 100$

$x_1, x_2, x_3, x_4 >= 0$

Matrix form

Max Z = (4, 5, 9, 11, 0, 0, 0) $(x_{1,}\ x_2, x_{3,} x_4\ x_{5,}\ x_6, x_7)$

STC

$$\left\{\begin{matrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 7 & 5 & 3 & 2 & 0 & 1 & 0 \\ 3 & 5 & 10 & 15 & 0 & 0 & 1 \end{matrix}\right\} \left\{\begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{matrix}\right\} = \left\{\begin{matrix} 7 \\ 120 \\ 100 \end{matrix}\right\}$$

$$x_{1,}\ x_2, x_{3,} x_4\, x_5, x_{6,} x_7 >= 0$$

Starting table 4 5 9 11 0 0 0

Basic variable	Profit / unit	solution	x_1	x_2	$x_{3,}$	x_4	x_5	x_6	x_7	Min ratio
x_5	0	7	1	1	1	1	1	0	0	7/1
x_6	0	120	7	5	3	2	0	1	0	120/2
x_7	0	100	3	5	10	15	0	0	1	100/15
	MaxZ = 0x7+0x120 +0x100		-4	-5	-9	-11	0	0	0	

BIG M Method or Methods of Penalties

Whenever the objective function is MinZ and when all or some of the constraints are of >= type or = type. We introduce surplus variable and a artificial variable to LHS of the constraint when it is necessary to complete the identity matrix I.

The general practice is to assign the letter M as the cost in a minimization problem, and – M as the profit in the maximization problem with assumption that M is a very large positive number to the artificial variables in the objective function.

The method of solving a LPP in which a high penalty cost has been assigned to the artificial variables is known as the method of penalties or BIG m Method.

Procedures

Step1: At any iteration of the usual simplex method can arise any one of the following three cases:

Case a) if there is no vector corresponding to some artificial variable in the solution column in such case, we proceed to step 2.

Case b) if at least one vector corresponding to some artificial variable, in the basis is basic variable column at the zero level i.e., corresponding entry in solution column is zero and the co-efficient of m in each net evaluation Z_j - C_j is non negative.

In such case, the current basic feasible solution is a degenerate one.
If this is a case when an optimum solution. The given LPP includes an artificial basic variable and an optimum basic feasible solution does not exist.

Case c) if at least one artificial vector is in the basis Y_b but, not at zero level i.e., the corresponding entry in X_b is non zero. Also co-efficient of M in each net evaluation $Z_j - C_j$ is non negative.

In the case, the given LPP does not possess an optimum basic feasible solution. Since, M is involved in the objective function. In such case, the given problem has a pseudo optimum basic feasible solution.

Step 2: application of simplex method is continued until either an optimum basic feasible solution is obtained or there is an indication of the existence of an unbounded solution to the given LPP.

Note: while applying simplex method, whenever a vector corresponding to some artificial variable happens to leave the basis, we drop that vector and omit all the entries corresponding to that vector from the simplex table.

Problem 1:
Solve the give LPP by BIG M Method
Max Z =$3x_1 - x_2$
STC
$2x_1 + x_2 >= 2$
$x_1 + 3x_2 <= 3$
$8x_2 <= 4$
$x_1, x_2 >= 0$

Since, the given problem is max z and the I constraint is >= type we introduce surplus variable as $x_3 >= 0$, and a artificial variable as $x_4 >= 0$, the II & III constraint are of <= type, so we introduce $x_5 >= 0$, $x_6 >= 0$.

The standard of the given LPP as follow as.

Max Z =$3x_1 - x_2 + 0x_3 - M\, x_4 + 0\, x_5 + 0x_6$

STC

$2x_1 + x_2 - x_3 + x_4 \quad = 2$

$x_1 + 3x_2 + x_5 \quad = 3$

$x_2 + x_6 \quad = 4$

$x_1, x_2, x_3,\ x_4\ x_5, x_6 >= 0$

The Matrix form

Max Z= (3,-1, 0,-M, 0, 0) $(x_1, x_2,\ x_3,\ x_4\ x_5, x_6)$

STC

$$\left\{\begin{array}{cccccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right\} \left\{\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{array}\right\} = \left\{\begin{array}{c} 2 \\ 3 \\ 4 \end{array}\right\}$$

$x_1, x_2, x_3,\ x_4\ x_5, x_6 >= 0$

$$\left\{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right\}$$

$(x_4\ x_5, x_6)$ canonical system.

The basic variable and their obvious solution

Starting table: 3 -1 0 -M 0 0

Basic Variable	Profit / unit	solution	x_1	x_2	x_3	x_4	x_5	x_6	Min ratios
x_4	-M	2	2	1	-1	1	0	0	2/2=1
x_5	0	3	1	3	0	0	1	0	3/1=3
x_6	0	4	0	8	0	0	0	1	4/0=∞
	MaxZ= -mx2+0x3 +0x4=-2m		=-mx2+0x1+0x4-3= -2m-3	=-mx1+0x3+0x0-(-1)= -m+1	=-mx-1+0x0+0x0-0= m	=-mx1+0x0+0x0-)-m)=0	=-mx0+0x1+0x0-0-0=0	=-mx0+0x0+0x1-0=0	

3 -1 0 -M 0 0

Basic Variable	Profit / unit	solution	x_1	x_2	x_3	x_4	x_5	x_6	Min ratios
x_1	3	1	1	½	-1/2	1/2	0	0	1/-1/2=-2
x_5	0	2	0	5/2	1/2	-1/2	1	0	2/1/2=4
x_6	0	4	0	1	0	0	0	1	4/0=∞
	MaxZ=3x1+0x2+0x4=3		=3x1+0x0+0x0 – 3	3x1/2+0x5/2+0x1-(-1)	=3x-1/2+0x1/2+0x0-0	=3x1/2+ox-1/2+0x0 – (-M)	=3x0+0x1+0x0-0	=3x0+0x0+0x1-0	
			0	5/2	-3/2	3/2+m	0	0	

New Number for Key row is

New no for key row =old element/pivot element

Soln= 2/2=1, x_1 =2/2=1, x_2 =1/2, x_3 = -1/2, x_4 =1/2, x_5 =0/2=0, x_6 =0/2=0

Other than key row new no. = old element - PCE*NPRE

Soln= 3-1*1=2, x_1 =1-1*1=0, x_2 =3-1*1/2=5/2, x_3 =0-1*-1/2=1/2, x_4 =0-1*1/2=-1/2, x_5 =1-1*0=1, x_6 =0-1*0=0

For x_6 new no are

Soln= 4-0*1=4, x_1 =0-0*1=0, x_2 =8-0*1/2=8, x_3 =0-0*-1/2=0, x_4 =0-0*1/2=0, x_5 =0-0*0=0, x_6 =1-0*0=1

			3	-1	0	-M	0	0	
Basic Variable	Profit / unit	solution	x_1	x_2	x_3	x_4	x_5	x_6	Min ratios
x_1	3	3	1	3	0	0	1	0	
x_3	0	4	0	5	1	-1	2	0	
x_6	0	4	0	1	0	0	0	1	
	Max Z =3x3+0x4+0x4=9		=3x1+0x0+0x0-3	=3x3+0x5+0x1-(-1)	=3x0+0x1+0x0-0	3x0+0x-1+0x0-(-M)	=3x1+0x2+0x0-0	=0x3+0x0+0x1-0	
			0	10	0	M	3	0	

New Number for Key row is

New no for key row =old element/pivot element (x_3)

Soln= 2/1/2=4, x_1 =0/1/2=0, x_2 =5/2/1/2=5, x_3 = 1/2/1/2=1, x_4 =-1/2/1/2=-1, x_5 =1/1/2=2, x_6 =0/1/2=0

Other than key row new no. x_1 row = old element – PCE *NPRE

Soln = 1-(-1/2)*4=3, x_1 =1+1/2*0=1, x_2 =1/2+1/2*5=3, x_3 =-1/2-(-1/2)*1=0, x_4 =1/2+1/2*-1=0, x_5 =0+1/2*2=1, x_6 =0-(-1/2)*0=0

For x_6 new no are

Soln= 4-0*4=4, x_1 =0-0*0=0, x_2 =1-0*5=1, x_3 =0-0*1=0, x_4 =0-0*-1=0, x_5 =0-0*0=0, x_6 =1-0*0=1

Since all the NER is +ve and at zero level which is optimum.

This problem is of case A, which means no artificial vector appears at the optimal table and therefore, the given problem as attained the optimality.

$x_1=3, x_2=0, x_3=4, x_4=0, x_5=0x_6=4$ substitute this values in the objection function.

Max Z =3x3 - 0 +0x4 –Mx0+0x0+ 0x4 = 9

Problem:2

Solve the given problem by charnes penalty method

Max $Z = 3x_1 + 2x_2$

STC

$2x_1 + x_2 \quad <= 2$

$3x_1 + 4x_2 \quad >= 12$

$x_1, x_2 >= 0$

Solution:

Since, the given problem is max z

The I constraint is <= type we introduce slack variable as $x_3 >= 0$ and the II constraint is of >= type we introduce surplus variable as $x_4 >= 0$ and a artificial variable as $x_5 >= 0$. In the constraint the value of artificial variable will be 1 and in the objective function is –M.

Therefore, the standard form of the problem

Max $Z = 3x_1 + 2x_2 + 0x_3 + 0x_4 - Mx_5$

STC

$2x_1 + x_2 + x_3 = 2$

$3x_1 + 4x_2 - x_4 + x_5 = 12$

$x_1, x_2, x_3, x_4 x_5 >= 0$

The Matrix form

Max Z= $(3, 2, 0, 0, -M)\ (x_1, x_2, x_3, x_4 x_5)$

STC

$$\begin{Bmatrix} 2 & 1 & 1 & 0 & 0 \\ 3 & 4 & 0 & -1 & 1 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 12 \end{Bmatrix}$$

$x_1, x_2, x_3, x_4 x_5 >= 0$

$$\left\{\begin{matrix}1 & 0\\ 0 & 1\end{matrix}\right\}$$

Canonical system. The basic variable ($x_{3,}x_5$) and their obvious solution (2,12)

Starting table: 3 2 0 0 -M

Basic Variable	Profit / unit	solution	x_1	x_2	x_3	x_4	x_5	Min ratios
x_3	0	2	2	1	1	0	0	2/1=2
x_5	-M	12	3	4	0	-1	1	12/4=3
	Max Z=0x2+-mx12		=0x2+-mx3-3	=0x1+-mx4-2	=0x1+-mx0-0	=0 x 0 +- mx-1-0	=0x0+-m x1-(-m)	
	-12m		-3m-3	-4m-2	0	m	0	

Starting table: 3 2 0 0 -M

Basic Variable	Profit / unit	solution	x_1	x_2	x_3	x_4	x_5	Min ratios
x_2	2	2	2	1	1	0	0	
x_5	-m	4	-5	0	-4	-1	1	
	Maxz= 2x2+-mx4		5m+1	0	4m+2	m	0	

Since, the co-efficient of m in each $Z_j - C_j$ is non-negative and an artificial vector appeared in the basis, not at zero level.

Then, given LPP does not possess an optimum basic feasible solution. So, that is existence of pseudo optimum BFS to the given LPP.

Problem:3

Solve the given problem by charnes penalty method

Max $Z = 3x_1 + 2x_2 + 3x_3$

STC

$2x_1 + x_2 + x_3 \quad <= 2$

$3x_1 + 4x_2 + 2x_3 \quad >= 8$

$x_1, x_2, x_3 >= 0$

Solution:

Since, the given problem is max z

The I constraint is <= type we introduce slack variable as $x_4 >= 0$ and the II constraint is of >= type we introduce surplus variable as $x_5 >= 0$ and a artificial variable as $x_6 >= 0$.

In the constraint the value of artificial variable will be 1 and in the objective function is –M. Therefore, the standard form of the problem.

Max $Z = 3x_1 + 2x_2 + 3x_3 + 0\,x_4 + 0\,x_5 - M\,x_6$

STC

$2x_1 + x_2 + x_3 + x_4 = 2$

$3x_1 + 4x_2 - x_5 + x_6 = 8$

$x_1, x_2, x_3, x_4\ x_5\ x_6 >= 0$

The Matrix form

Max $Z = (3,2,3,0,0,-M)\,(x_1, x_2, x_3, x_4\ x_5, x_6)$

STC

$$\begin{Bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 & -1 & 1 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 8 \end{Bmatrix} \qquad \begin{Bmatrix} 1 & 0 \\ 0 & 1 \end{Bmatrix}$$

Canonical system. The basic variable (x_4, x_6) and their obvious solution (2, 8)

$x_1, x_2, x_3, x_4\ x_5\ x_6 >= 0$

Starting table: 3 2 3 0 0 -M

Basic Variable	Profit / unit	solution	x_1	x_2	x_3	x_4	x_5	x_6	Min ratios
X_4	0	2	2	1	1	1	0	0	2/1=2
x_6	-M	8	3	4	2	0	-1	1	8/4=2
			=0x2+-mx3-3	=0x1+-mx4-2	=0x1+-mx2-3	=0 x 1 +- mx0-0	=0x0+-m x-1- 0	=0x0+-mx1-(-m)	
			-3m-3	-4m-2	-2m-3	0	m	0	

Starting table: 3 2 0 0 0 -M

Basic Variable	Profit / unit	solution	x_1	x_2	x_3	x_4	x_5	x_6	Min ratios
x_2	2	2	2	1	1	1	0	0	
x_6	-m	0	-5	0	-2	-4	-1	1	
	maxZ=2x2+-mx0=4		5m+1	1	2m+1	4m+2	m	0	

Since, the co-efficient of m in each $Z_j – C_j$ is non-negative and an artificial vector appears in the basis, at zero level.

Then indicates the existence of an optimum basic feasible solution to the given LPP. Thus, an optimum basic feasible solution to the given LPP.

$x_1 =0$	$x_2 =2$	$x_3 =0$	$x_4 =0$	$x_5 =0$	$x_6 =0$

Max $Z =3x_1 +2x_2 +3x_3 +0\ x_4+0\ x_5-M\ x_6$

=3*0+2*2+3*0+0*0+0*0-M*0

=4

Problem No : 4

Solve the given problem by charnes penalty method

Max $Z = 3x_1 + 2x_2 + 4x_3$

STC

$2x_1 + 5x_2 + x_3 = 12$

$3x_1 + 4x_2 = 11$

$x_1, x_2, x_3 >= 0$

Problem No 5

Solve the given LPP

Min $Z = 4x_1 + x_2$

STC

$3x_1 + 2x_2 = 3$

$4x_1 + 3x_2 >= 6$

$x_1 + 2x_2 <= 4$

$x_1, x_2 >= 0$

The I constraint is already in standard form, only artificial variable is added as $x_3 >= 0$ and II constraint is of >= type we introduce $x_4 >= 0$ as surplus variable and an artificial variable as $x_5 >= 0$ and third constraint is of <= type we introduce $x_6 >= 0$ as slack variable.

Then, the given LPP is in standard form

Min $Z = 4x_1 + x_2 + Mx_3 + 0x_4 + Mx_5 + 0x_6$ converting MinZ= -Z

or

Max $Z = -4x_1 - x_2 - Mx_3 - 0x_4 - Mx_5 - 0x_6$

STC

$3x_1 + 2x_2 + x_3 = 3$

$4x_1 + 3x_2 - x_4 + x_5 = 6$

$x_1 + 2x_2 + x_6 = 4$

$x_1, x_2, x_3, x_4, x_5, x_6 >= 0$

The matrix form

Max Z = (-4, -1, -M, 0, -M, 0) $(x_1, x_2, x_3, x_4, x_5, x_6)$

STC

$$\begin{Bmatrix} 3 & 2 & 1 & 0 & 0 & 0 \\ 4 & 3 & 0 & -1 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{Bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{Bmatrix} 3 \\ 6 \\ 4 \end{Bmatrix}$$

$x_1, x_2, x_3, x_4, x_5, x_6 >= 0$

Starting table: - 4 -1 -m 0 -m 0

Basic Variable	Profit / unit	solution	x_1	x_2	x_3	x_4	x_5	x_6	Min ratios
X_3	-m	3	3	2	1	0	0	0	3/3=1
x_5	-m	6	4	3	0	-1	1	0	6/4=1.5
x_6	0	4	1	2	0	0	0	1	4/1
	Max z= -9m		-7m+4	-5m+1	0	m	0	0	

Starting table: - 4 -1 -m 0 -m 0

Basic Variable	Profit / unit	solution	x_1	x_2	x_3	x_4	x_5	x_6	Min ratios
X_1	-4	1	1	2/3	1/3	0	0	0	3/2
x_5	-m	2	0	1/3	-4/3	-1	1	0	6
x_6	0	3	0	4/3	-1/3	0	0	1	9/4
	Max z= -4-2m		0	.-5-m /3	4m-4 /3	m	0	0	

Starting table: - 4 -1 -m 0 -m 0

Basic Variable	Profit / unit	solution	x_1	x_2	x_3	x_4	x_5	x_6	Min ratios
X_2	-1	3/2	3/2	1	1/2	0	0	0	
x_5	-m	3/2	-1/2	0	-2	-1	1	0	
x_6	0	1	-2	0	-3	0	0	1	
	Max z= -3/2-3/2m		5+m/2	0	-1/2+2m	m	0	0	

Problem 6

Solve the given problem by BIG M Method

Min Z = $-3x_1 + x_2 + x_3$

STC

$x_1 + -2x_2 + x_3 \quad <= 11$

$-4x_1 + x_2 + 2x_3 \quad > = 3$

$24x_1 - x_3 \quad = -1$

$x_1, x_2, x_3 \quad >= 0$

Solution:

The I constraint is of <= type so add $x_4 >= 0$ as slack variable, II constraint is of >= type so we introduce $x_5 >= 0$ as surplus variable and $x_6 >= 0$ as artificial variable and III constraint is already a standard form and right side is –ve. So it should be multiplied by -1 both sides of the equation and $x_7 >= 0$ as artificial variable and the standard form of the given LPP is written below.

Min Z = $-3x_1 + x_2 + x_3 + 0x_4 + m\,x_5 + x_6 + m\,x_7$

STC

$x_1 + -2x_2 + x_3 + x_4 \quad = 11$

$-4x_1 + x_2 + 2x_3 + x_5 + x_6 \quad = 3$

$-2x_1 + x_3 + x_7 \quad = 1$

$x_1, x_2, x_3, x_4\ x_5\ x_6\ x_7 \quad >= 0$

The matrix form

Min z = (-3,1,1,0,M,1,M) ($x_{1,}$ $x_{2,}$ $x_{3,}$ x_4 x_5 x_6 x_7)

STC

$$\begin{Bmatrix} 1 & -2 & 1 & 1 & 0 & 0 & 0 \\ -4 & 1 & 2 & 0 & -1 & 1 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 11 \\ 3 \\ 1 \end{Bmatrix}$$

$x_{1,}$ $x_{2,}$ $x_{3,}$ x_4 x_5 x_6 x_7 >= 0

Starting table: -3 1 1 0 m 1 m

Basic Variable	Profit / unit	solution	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Min ratios
X_4	0	11	1	-2	1	1	0	0	0	11/1
X_6	M	3	-4	1	2	0	-1	1	0	3/2
x_7	M	1	-2	0	1	0	0	0	1	1/1
	MaxZ=0x11+3xM+ 1xm=4m		-3+6m	1-m	1-3m	0	2m	1-m	0	

NER= C_j –(P/U* x_j) (j=12….n)

x_1= -3-(0x1+mx-4+-2m) = -3+6m

x_2= 1- (0x-2+mx1+0xm) = 1-m

x_3= 1- (0x1+mx2+1xm) = 1-3m

x_4= 0- (0x1+mx0+0xm) = 0

x_5= m- (0x0+mx-1+0xm) = 2m

x_6= 1- (0x0+mx1+0xm) = 1-m

x_7= m- (0x0+mx0+1xm) = 0

First table: -3 1 1 0 m 1 m

Basic Variable	Profit / unit	solution	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Min ratios
X_4	0	10	3	-2	0	1	0	0	-1	10/-2= -ve
X_6	M	1	0	1	0	0	-1	1	-2	1/1=1
X_3	1	1	-2	0	1	0	0	0	1	1/0=0
	Min Z= 0x10 + mx1 +1x1 =1+m		-3-(0x3+mx0+1x-2) = -1	1-(0x-2+mx1+1x0)=1-m	1-(0x0+mx0+1x1) = 0	0-(0x1+mx0+1x0)= 0	m-(0x0+mx-1 + 1x0)= 2m	=1-(0x0+mx1+1x0) = 1-m	m -(0x-1+mx-2+1x1) = 3m-1	

The values of key row remain the same because 1 is the key element

New nos for other than key rows i.e X_4

Soln= 11-1*1=10

x_1= 1-1*-2 =3

x_2= -2-1*0=-2

x_3=1-1*1=0

x_4=1-1*0=1

x_5=0-1*0=0

x_6=0-1*0=0

x_7=0-1*1=-1

X_5,Soln= 3-2*1=1

x_1= -4-2*-2= 0

x_2= 1-2*0=1

x_3= 2-2*1=0

x_4= 0 -2*0=0

x_5= -1-2*0=-1

x_6= 1-2*0=1

x_7=0-2*1=-2

Third table: -3 1 1 0 m 1 m

Basic Variable	Profit / unit	Solution	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Min ratios
X_4	0	12	3	0	0	1	2	-2	-5	12/3
X_2	1	1	0	1	0	0	1	-1	-2	1/0
X_3	1	1	-2	0	1	0	0	0	1	1/-2
			-1	0	0	0	m-1	1	m+1	

Fourth table: -3 1 1 0 m 1 m

Basic Variable	Profit / unit	Solution	x_1	x_2	x_3	x_4	x_5	x_6	x_7	Min ratios
X_1	-3	4	1	0	0	1/3	2/3	-2/3	-5/3	
X_2	1	1	0	1	0	0	1	-1	-2	
X_3	1	9	0	0	1	2/3	4/3	-4/5	-7/3	
			0	0	0	1/3	m-1/3	1/3	m-2/3	

Table 4 is optimal and the value of the decision variable is

$x_1 =4$, $x_2 =1$, $x_3 =9$, $x_4 =0$, $x_5 =0$,$x_6 =0$ $x_7 = 0$

Min Z = -3*4 +1*1 + 1*9 + 0*0 +0*0+ 0*0 +0*)

=-12+1+9

= -2

Duality Theory

The linear programming model we develop for a situation is referred to as the **primal** problem. The dual problem can be derived directly from the primal problem.

The standard form has three properties

1. All the constraints are equations (with nonnegative right-hand side)
2. All the variables are nonnegative
3. The sense of optimization may be maximization of minimization.

Comparing the primal and the dual problems, we observe the following relationships.

1. The objective function coefficients of the primal problem have become the right-hand side constants of the dual. Similarly, the right-hand side constants of the primal have become the cost coefficients of the dual
2. The inequalities have been reversed in the constraints
3. The objective is changed from maximization in primal to minimization in dual
4. Each column in the primal correspondence to a constraint (row) in the dual. Thus, the number of dual constraints is equal to the number of primal variables.
5. Each constraint (row) in the primal corresponds to a column in the dual. Hence, there is one dual variable for every primal constraints
6. The dual of the dual is the primal problem.
7. if the primal constraints >= the dual constraints will be <= & vice versa

Duality is an extremely important and interesting feature of linear programming. The various useful aspects of this property are

i) If the primal problem contains a large no of rows constraints and smaller no of columns variables computational procedure can be considerably reduced by converting it into dual and then solving it.

ii) It gives additional information as to how the optimal solution changes as a result of the changes in the coefficients and the formulation of its problem.

iii) Calculations of the dual checks the accuracy of the primal solution

iv) This indicates that fairly close relationships exist between LP and theory of games

Primal Problem	Dual Problem
Max Z = CX STC Ax<=b x>=0	Min Z = bY STC Ay>=c y>=0
Max Z = CX STC ex: Ax=b x1+x2<=2 x>=0 -x1-x2<=-2	Min Z = bY STC Ay>=c Y is unrestricted in sign
Max Z = CX STC Ax=b X is unrestricted in sign	Min Z = bY STC Ay=c y is unrestricted in sign
Max Z = CX STC A1x<=b1 A2x=b2 x>=0	Min Z = b1Y1+b2y2 STC A1y1+A2y2>=c Y1>=0 y2 is unrestricted in sign

Note: it is not necessary that only the Max problem be taken as the primal problem we can as well consider the minimization LPP as the primal

Formulation of dual to the primal problem

Problem no 1

Write the dual of the primal problem

Max Z = $3x_1 + 5x_2$

STC

$2x_1 + 6x_2 \quad <= 50$

$3x_1 + 2x_2 \quad < = 35$

$5x_1 - 3x_2 \quad <= 10$

$x_2 \quad <= 20$

$x_1, x_2, x_3 \quad >= 0$

To write dual for the above primal, since the primal as 4 constraints the dual will have 4 variables as y_1 y_2 y_3 y_4 then dual for the primal will be as follows.

Min Z = $50y_1 + 35y_2 + 10y_3 + 20y_4$

STC

$2\ y_1 + 3y_2 + 5y_3 + 0y_4 >= 3$

$6y_1 + 2y_2 - 3y_3 + y_4 \quad > = 5$

$y_1, y_2, y_3, y_4 \quad >= 0$

it can be observed from the dual problem has less no constraint as compared to the primal problem (in case of primal they are 4 and in case of dual they are 2) which requires less work and effort to solve it

Problem 2

Construct the dual of the given problem

Min Z = $3x_1 - 2x_2 + 4x_3$

STC

$3x_1 + 5x_2 + 4\ x_3 \quad >= 7$

$6x_1 + x_2 + 3x_3 \quad > = 4$

$7x_1 - 2\ x_3 - x_3 \quad < = 10$

$x_1 - 2x_2 + 5\ x_3 \quad >= 3$

$4x_1 + 7x_2 - 2\ x_3 \quad >= 2$

$x_1, x_2, x_3 \quad >= 0$

Solution:

The given problem is minimization type and all the constraint should be >= type. In the given problem third constraint is <= type so we must convert constraint to >= type by multiply both side of the constraint by -1, we get

$-7x_1 +2 x_3 +x_3 \quad >= - 10$

Then given the problem can be written restated

Min $Z = 3x_1 -2x_2 + 4x_3$

STC

$3x_1 +5x_2+4 x_3 \quad >=7$

$6x_1 +x_2+ 3x_3 \quad >= 4$

$-7x_1 +2 x_2 +x_3 \quad >= -10$

$x_1 -2x_2+5 x_3 \quad >= 3$

$4x_1 +7x_2-2 x_3 \quad >= 2$

$x_1, x_2, x_3 \quad >= 0$

Then dual of the given problem is as follows and the dual variables are $y_1 ,y_2 ,y_3,y_4\ y_4$

Max $Z = 7y_1 +4y_2 -10y_3+3y_4 +2y_5$

STC

$3 y_1 +6y_2 -7y_3+y_4+4y_4 \quad >= 3$

$5y_1 +y_2 +2y_3-2y_4 +7y_4 \quad >= -2$

$4y_1 +3y_2 +y_3+5y_4 -2y_4 \quad >= 4$

$y_1 ,y_2 ,y_3,y_4 ,y_5 \quad >= 0$

Problem 3

Obtain the dual problem of the following LPP

Max $Z = 2x_1 + 5x_2 + 6x_3$

STC

$5x_1 + 6x_2 - 4x_3 \quad <= 3$

$-2x_1 + x_2 + 4x_3 \quad < = 4$

$x_1 - 5x_2 + 3x_3 \quad < = 1$

$-3x_1 - 3x_2 + 7x_3 \quad <= 6$

$x_1, x_2, x_3 \quad >= 0$

Also, verify that the dual of the dual problem is the primal problem

The given primal can be restated in the matrix form

Max $Z = (2,5,6)\ (x_1, x_2, x_3)$

STC

$$\begin{Bmatrix} 5 & 6 & -4 \\ -2 & 1 & 4 \\ 1 & -5 & 3 \\ -3 & -3 & 7 \end{Bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} <= \begin{Bmatrix} 3 \\ 4 \\ 1 \\ 6 \end{Bmatrix}$$

$x_1, x_2, x_3 \quad >= 0$

The dual of the above primal can be written as and the dual variables are (y_1, y_2, y_3, y_4)

Min $w = (y_1, y_2, y_3, y_4)\ (3.4.1.6)$

STC

$$\begin{Bmatrix} 5 & -2 & 1 & -3 \\ 6 & 1 & -5 & 3 \\ -4 & 4 & 3 & 7 \end{Bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 5 \\ 6 \end{Bmatrix}$$

$y_1, y_2, y_3, y_4 >= 0$

Now, we can restate this dual problem as follows

Max w = (-3.-4.-1.-6) (y_1, y_2, y_3, y_4)

$$\begin{array}{r}\\ \\ (-1)\end{array}\left\{\begin{matrix} 5 & -2 & 1 & -3 \\ 6 & 1 & -5 & -3 \\ -4 & 4 & 3 & -7 \end{matrix}\right\}\left\{\begin{matrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{matrix}\right\} = \left\{\begin{matrix} -2 \\ -5 \\ -6 \end{matrix}\right\}$$

$y_1, y_2, y_3, y_4 >= 0$

MaxZ = (-1) Minz

The dual problem in this form looks like the primal problem and this we may write down the dual of this dual problem .

MinZ= (-2,-5, -6) (x_1, x_2, x_3)

STC

$$\begin{array}{r}\\ \\ \\ (-1)\end{array}\left\{\begin{matrix} 5 & 6 & -4 \\ -2 & 1 & 4 \\ 1 & -5 & 3 \\ -3 & -3 & 7 \end{matrix}\right\}\left\{\begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}\right\} <= \left\{\begin{matrix} -3 \\ -4 \\ -1 \\ -6 \end{matrix}\right\}$$

$x_1, x_2, x_3 >= 0$

or Max z = (2 5 6) (x_1, x_2, x_3)

STC

$$\left\{\begin{matrix} 5 & 6 & -4 \\ -2 & 1 & 4 \\ 1 & -5 & 3 \\ -3 & -3 & 7 \end{matrix}\right\}\left\{\begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}\right\} <= \left\{\begin{matrix} -3 \\ -4 \\ -1 \\ -6 \end{matrix}\right\}$$

$x_1, x_2, x_3 >= 0$

Problem 4

Construct the dual of the primal problem

Max $Z = 3x_1 + 17x_2 + 4x_3$

STC

$x_1 - x_2 + x_3 \quad >= 3$

$-3x_1 + 2x_3 \quad < = 1$

$x_1, x_2, x_3 \quad >= 0$

Problem 5

Construct the dual of the following primal

Max $Z = x_1 - 2x_2 + 3x_3$

STC

$-2x_1 + x_2 + 3x_3 \quad = 2$

$2x_1 + 3x_2 + 4x_3 \quad = 1$

$x_1, x_2, x_3 \quad >= 0$

The given problem is 2nd variety in the duality and it is in the form of Max Z, then the constraint in the standard form can be written as

$-2x_1 + x_2 + 3x_3 \quad <= 2$

$2x_1 - x_2 - 3x_3 \quad <= -2$

and

$2x_1 + 3x_2 + 4x_3 \quad <= 1$

$-2x_1 - 3x_2 - 4x_3 \quad <= -1$

Then, given primal can restated as

Max $Z = x_1 - 2x_2 + 3x_3$

STC

$-2x_1 + x_2 + 3x_3 \quad <= 2$

$2x_1 - x_2 - 3x_3 \quad <= -2$

$2x_1 + 3x_2 + 4x_3 \quad <= 1$

$-2x_1 - 3x_2 - 4x_3 \quad <= -1$

$x_1, x_2, x_3 \quad >= 0$

The dual of the above primal is written as and its dual variables are y_1, y_2, y_3

$\text{Min } W = 2y_1 - 2y_2 + y_3 - y_4$
STC

$-2y_1 + 2y_2 + 2y_3 - 2y_4 >= 1$

$y_1 - y_2 + 3y_3 - 3y_4 >= -2$

$3y_1 - 3y_2 + 4y_3 - 4y_4 >= 3$

$y_1, y_2, y_3\ y_4 >= 0$

This can be written

$\text{Min } W = 2y_1 + y_2$

STC

$-2y_1 + 2y_2 >= 1$

$y_1 + 3y_2 >= -2$

$3y_1 + 4y_2 >= 3$

y_1, y_2 are unrestricted in sign

Problem 6

Write the dual of the given LPP

$\text{Max } Z = 3x_1 + x_2 + 2x_3 - 2x_4$

STC

$2x_1 - x_2 + 3x_3 + x_4 = 1$

$x_1 + x_2 - x_3 + x_4 = -3$

$x_1, x_2 >= 0$, x_3 & x_4 are unrestricted in sign

Problem 7

Write the dual of the following LPP

$\text{Min } Z = 2x_1 + 3x_2 + 4x_3$

STC

$2x_1 + 3x_2 + 5x_3 >= 2$

$3x_1 + x_2 + 7x_3 = 2$

$x_1 + 4x_2 + x_3 <= 5$

$x_1, x_2, >= 0$ x_3 are is unrestricted in sign

Use duality to solve the following LPP

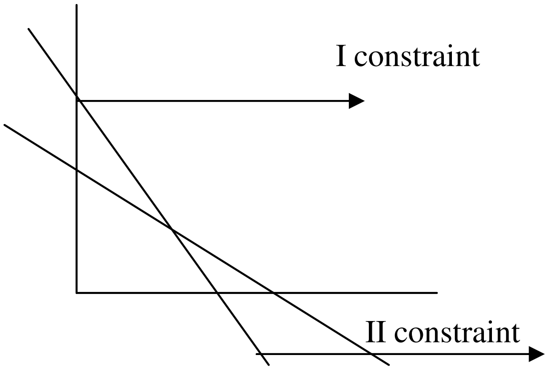
Max $Z = 3x_1 + 2x_2$

STC

$2x_1 + x_2 <= 5$ the points are (5/2,5)

$x_1 + x_2 <= 3$ (3,3)

$x_1, x_2 >= 0$



Use duality concept the dual of the above primal as follows

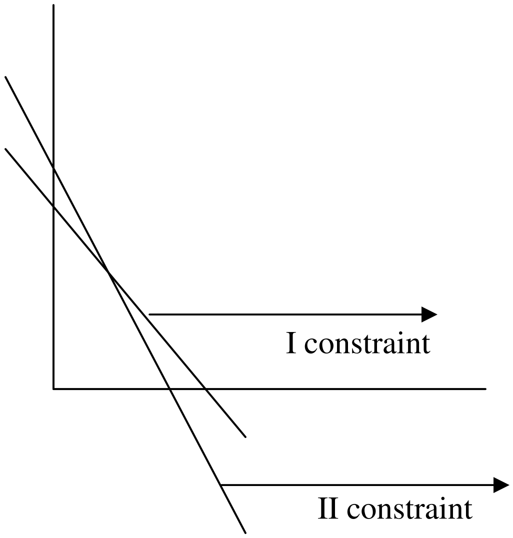
Min $Z = 5y_1 + 3y_2$

STC

$2y_1 + y_2 >= 3$ (3/2,3)

$y_1 + y_2 >= 2$ (2,2)

$y_1, y_2 >= 0$



Using duality theory solve the following LPP by simplex method

Min $Z = 4x_1 + 3x_2 + 6x_3$

STC

$x_1 + x_3 >= 2$

$x_2 + x_3 >= 5$

$x_1, x_2, x_3 >= 0$

the dual of the above primal

Max $Z = 2y_1 + 5y_2$

STC

$y_1 <= 4$

$y_2 <= 3$

$y_1 + y_2 <= 6$

$y_1, y_2 >= 0$

Using duality theory solve the following LPP by simplex method

Max $Z = x_1 + 6x_2$

STC

$x_1 + x_2 \quad >= 2$

$x_1 + x_2 \quad <= 3$

$x_1, x_2 \quad >= 0$

The given problem is max Z so all the constraint should be <= type. In this problem I constraint is of >= type so we converting by multiply b -1 both sides of the constraint the resulting will be as

$-x_1 - x_2 \quad <= -2$

Then, the given LPP is restated as

Max $Z = x_1 + 6x_2$

STC

$-x_1 - x_2 \quad <= -2$

$x_1 + x_2 \quad <= 3$

$x_1, x_2 \quad >= 0$

The dual of the above LPP will be as

Min $Z = 2y_1 + 3y_2$

STC

$-y_1 + y_2 \quad >= 1$

$-y_1 + y_2 \quad >= 6$

$x_1, x_2 \quad >= 0$

Solving the dual using Big M method

Dual simplex method

Dual simplex method applies to problems which start with dual feasible solns. The objective function may be either in the maximization form or in the minimization form. After introducing the slack variables, if any right-hand side element is –ve and if the optimality condition is satisfied.

The problem can be solved by the dual simplex method

Procedure for dual simplex method

Step1) obtain an initial basic solution to the LPP and put the solution in the starting dual simplex table

Step2) test the nature of Z_j-C_j in the starting simplex table

a)if all Z_j-C_j and solution column are non-negative for all I and j, then an optimum basic feasible solution has been obtained.

b) if all Z_j-C_j are non negative and at least on basic variable in the solution column is negative go step 3

c) if at least one Z_j-C_j is –ve the method is not applicable to the given problem

Step 3) selects the most –ve in solution column

step4) test the nature of

a) if all x_{ij} are non negative the given problem does not exist any feasible solution
b) b) if at least one x_{ij} is –ve, compute the ratios Z_j-C_j / x_{ij} , x_{ij} >=0 chose the maximum of theses ratios

Step 5) test the new iterated dual simplex table for optimality

Repeat the procedure until either an optimum feasible solution has been obtained or there is an indication of the non existence of a feasible solution.

Use dual simplex method to solve the LPP

Min $Z = x_1 + x_2$

STC

$2x_1 + x_2 \quad >= 2$

$-x_1 - x_2 \quad >= 1$

$x_1, x_2 \quad >= 0$

Since, the given problem in min z form it is converted to max z and all the constraint is of >= type it should be converted to <= type by multiplying -1 on the both sides then given problem will be as follows

Max $Z = -x_1 - x_2$

STC

$-2x_1 - x_2 \quad <= -2$

$x_1 + x_2 \quad <= -1$

$x_1, x_2 \quad >= 0$

Now, introducing slack variable for the I and II constraint as $x_3 >= 0$, $x_4 >= 0$

The standard form as follows

Max $Z = -x_1 - x_2 + 0x_3 + 0x_4$

STC

$-2x_1 - x_2 + x_3 \quad = -2$

$x_1 + x_2 + x_4 \quad = -1$

$x_1, x_2, x_3\ x_4 \quad >= 0$

The matrix form

Max $Z = (-1,-1,0,0)\,(x_1, x_2, x_3\ x_4)$

STC

$$\begin{pmatrix} -2 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

$x_1, x_2, x_3\ x_4 \quad >= 0$

the identity matrix form are $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ the correspond variables $(x_3\ x_4)$ and their solutions are -2 -1 starting table -1 -1 0 0

Basic Variable	Profit / Unit	solution	x_1	x_2	x_3	x_4
x_3	0	-2	-2	-1	1	0
x_4	0	-1	1	1	0	1
	Max Z= 0*-2+0*-1		=0*-2+0*1+1	=0*-1+0*1+1	=0*1+0*0 -0	=0*0+0*1-0
	0		1	1	0	0

Ratios = NER / X_{1j}

$$= \left[1/-2,\ 1/-1,\ 0/1,\ 0/0 \right]$$

The max negative in the ratios are 1/-2

First table -1 -1 0 0

Basic Variable	Profit / Unit	solution	x_1	x_2	x_3	x_4
x_1	-1	1	1	0.5	-1/2	0
x_4	0	-2	0	0.5	1/2	1
	Max Z= -1*1=0*-2		=-1*1+0*0+1	=0.5*-1+0.5*0+1	=-1*-1/2+0*1/2 -0	=-1 * 0 + 0*1-0
	-1		0	0.5	0	0

New no for key row are

Soln = -2/-2 = 1, -2/-2 =1, -1/-2=0.5, - 1/2,0/-2

New no for other than key rows is x_4

Soln=-1-1*1=-2, 1-1*1=0, 1-1*0.5=0.5, 0-1*-1/2=1/2, 1-1*0=1

Since, all the values in NER is positive and in solution column one variable is –ve and that is selected as key row and to select key column or pivot column at least on variable in the row should be –ve but, no vector corresponding to that row is –ve and we cannot find the ratios and So we cannot select the key column and the given problem does not given any feasible solution to the LPP.

Use dual simplex table to solve the LPP

Min $Z = x_1 + 2x_2 + 3x_3$

STC

$x_1 - x_2 + x_3 \quad >= 4$

$x_1 + x_2 + 2x_3 \quad <= 8$

$x_2 - x_3 \quad >= 2$

$x_1, x_2, x_3 \quad >= 0$

the given problem is set for the requirement of dual simplex method

Max $Z = -x_1 - 2x_2 - 3x_3$

STC

$-x_1 + x_2 - x_3 \quad <= -4$

$x_1 + x_2 + 2x_3 \quad <= 8$

$-x_2 + x_3 \quad <= -2$

$x_1, x_2, x_3 \quad >= 0$

since, all the constraint are of <= type we introduce three slack variable for I II and III constraint as $x_4 >= 0$, $x_5 >= 0$, $x_6 >= 0$

the standard form

Max $Z = -x_1 - 2x_2 - 3x_3 + 0x_4 + 0x_5 + 0x_6$

STC

$-x_1 + x_2 - x_3 + x_4 \quad = -4$

$x_1 + x_2 + 2x_3 + x_5 \quad = 8$

$-x_2 + x_3 + x_6 \quad = -2$

$x_1, x_2, x_3\ x_4\ x_5\ x_6 \quad >= 0$

The matrix form

Max Z (-1,-2,-3,0,0,0) ($x_1, x_2, x_3\ x_4\ x_5\ x_6$)

STC

$$\begin{Bmatrix} -1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{Bmatrix} = \begin{Bmatrix} -4 \\ 8 \\ -2 \end{Bmatrix} \quad \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix}$$

The IBFS are $x_4\ x_5\ x_6$

$x_1, x_2, x_3\ x_4\ x_5\ x_6 >= 0$ and their obvious soln (-4,8,-2)

Starting table -1 -2 -3 0 0 0

Basic Variable	Profit / unit	soln	x_1	x_2	x_3	x_4	x_5	x_6
x_4	0	-4	-1	1	-1	1	0	0
x_5	0	8	1	1	2	0	1	0
x_6	0	-2	0	-1	1	0	0	1
	MaxZ =0*-4+0*8+0*-8		1	2	3	0	0	0

The ratios =(1/-1, 2/-1, 3/-1, 0/0, 0/0, 0/1)

Other than key row new nos are

x_5

soln = 8-1*4=4, x_1 =1-1*1=0, x_2 =1-1*-1=0, x_3 =2-1*1=1, x_4 =0-1*-1=1

x_5 = 1-1*0=1, x_6 =0-1*0=0

x_6

soln= -2-0*4=-2, all other value in the row remains the same because of corresponding value of pivot column element is zero

Starting table -1 -2 -3 0 0 0

Basic Variable	Profit / unit	soln	x_1	x_2	x_3	x_4	x_5	x_6
x_1	-1	4	1	-1	1	-1	0	0
x_5	0	4	0	0	1	1	1	0
x_6	0	-2	0	-1	1	0	0	1
	Max Z =		0	3	2	1	0	0

Ratios = (0/0, 3/-1,2/1,1/0,0/0,0/1)

For x_1 row

soln=4-(-1)*2=6, x_1 =1-(-1)*0=1, x_2 =-1-(-1)1=0, x_3 =1-(-1)-1=0, x_4 =-1-(-1)*0=-1, x_5 =0-(-1)*0=0, x_6 =0-(-1)*-1=-1

For x_5 row

soln =4 - 0*2=4, any thing multiplied by 0 is 0 so the value in this row remains same.

Starting table -1 -2 -3 0 0 0

Basic Variable	Profit / unit	soln	x_1	x_2	x_3	x_4	x_5	x_6
x_1	-1	6	1	0	0	-1	0	-1
x_5	0	4	0	0	1	1	1	0
x_2	-2	2	0	1	-1	0	0	-1
	Max Z =		0	0	5	1	0	3

Since all the NER and Solution column are non negative and the given problem as attained the optimum

Therefore, $x_1 = 6$, $x_2 = 2$, $x_3 = 0$, $x_4 = 0$, $x_5 = 4$, $x_6 = 0$

These values are substituted in the objective function

Max Z = -6*1 -2*2 -3*0 +0*0+ 0*4+ 0*0

Dual Read off Technique

Solve the following the LPP and read the solution for the real problem

Min Z = $2x_1 + x_2$

STC

$x_1 + x_2 \quad >= 15$

$x_1 - x_2 \quad <= 3$

$2x_1 + 5x_2 \quad >= 20$

$-x_1 + 3x_2 \quad >= 10$

$x_1, x_2 \quad >= 0$

To write the dual of the above primal all the constraint should be of >= type, since, II constraint is <= type we must convert it into >= type by -1, then, the given primal can be written as follows

Min Z = $2x_1 + x_2$

STC

$x_1 + x_2 \quad >= 15$

$-x_1 + x_2 \quad >= -3$

$2x_1 + 5x_2 \quad >= 20$

$-x_1 + 3x_2 \quad >= 10$

$x_1, x_2 \quad >= 0$

The dual for the above primal problem as follows

The dual variables (y_1 y_2 y_3 y_4)

Max Z = $15y_1 - 3y_2 + 20y_3 + 10y_4$

STC

$y_1 - y_2 + 2y_3 - y_4 \quad <= 2$ $\qquad y_5 >= 0$ as slack variable

$y_1 + y_2 + 5y_3 + 3y_4 <= 1$ $\qquad y_6 >= 0$ as slack variable

y_1 y_2 y_3 y_4 $>= 0$

Matrix form

Max Z = $(y_1\ y_2\ y_3\ y_4)\ (15,-3,20,10)$

STC

$$\left\{\begin{matrix} 1 & -1 & 2 & -1 & 1 & 0 \\ 1 & 1 & 5 & 3 & 0 & 1 \end{matrix}\right\} \left\{\begin{matrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{matrix}\right\} = \left\{\begin{matrix} 2 \\ 1 \end{matrix}\right\}$$

$y_1\ y_2\ y_3\ y_4\ y_5\ y_6 >= 0$

$\left\{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right\}$ is identity matrix which form the basic variable ($y_5\ y_6$) for the I table and it's solution is (2,1)

Starting table 15 -3 20 10 0 0

Basic Variable	Profit / unit	soln	y_1	y_2	y_3	y_4	y_5	y_6	Min ratio
y_5	0	2	1	-1	2	-1	1	0	2/2=1
y_6	0	1	1	1	5	3	0	1	1/5=
	Max Z = 0		-15	3	-20	-10	0	0	

Starting table 15 -3 20 10 0 0

Basic Variable	Profit / unit	soln	y_1	y_2	y_3	y_4	y_5	y_6	Min ratio
y_5	0	8/5	3/5	0	-10/5	1	-1	-2/5	8/3
y_3	20	1/5	1/5	1/5	1	3/5	0	1/5	1
	Max Z = 0		-11	7	0	2	0	20	

Starting table 15 -3 20 10 0 0

Basic Variable	Profit / unit	soln	y_1	y_2	y_3	y_4	y_5	y_6	Min ratio
y_5	0	∞	0	-2	-3/5	-4	1	∞	
y_1	15	1	1	1	5	3	0	1	
	Max Z = 0		0	18	55	35	0	15	

Surplus variable ⟷ (y_1 to y_4) Real Primal ⟷ (y_5 to y_6)

Since all the NER non negative the given problem as attained optimum

Sl. No.	Primal	Dual
1	$X_1 = 0$	$Y_1 = 1$
2	$X_2 = 15$	$Y_2 = 0$
3	$X_3 = 0$	$Y_3 = 0$
4	$X_4 = 18$	$Y_4 = 0$
5	$X_5 = 55$	$y_5 = \infty$
6	$x_6 = 35$	$Y_6 = 0$

Reference Books:

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3. **D.S.Hira**, Operation Research, S.Chand & Company Ltd., New Delhi, 2004
4.