

Section 2.1 – Solving Linear Programming Problems

There are times when we want to know the maximum or minimum value of a function, subject to certain conditions. An **objective function** is a linear function in two or more variables that is to be optimized (maximized or minimized).

Linear programming problems are applications of linear inequalities, which were covered in Section 1.4. A **linear programming** problem consists of an objective function to be optimized subject to a system of constraints. The **constraints** are a system of linear inequalities that represent certain restrictions in the problem.

The problems in this section contain no more than two variables, and we will therefore be able to solve them graphically in the xy -plane. Recall that the solution set to a system of inequalities is the region that satisfies all inequalities in the system. In linear programming problems, this region is called the **feasible set**, and it represents all possible solutions to the problem. Each vertex of the feasible set is known as a **corner point**. The **optimal solution** is the point that maximizes or minimizes the objective function, and the **optimal value** is the maximum or minimum value of the function.

The context of a problem determines whether we want to know the objective function's maximum or the minimum value. If a linear programming problem represents the amount of packaging material used by a company for their products, then a minimum amount of material would be desired. If a linear programming problem represents a company's profits, then a maximum amount of profit is desired. In most of the examples in this section, both the maximum and minimum will be found.

Fundamental Theorem of Linear Programming

To solve a linear programming problem, we first need to know the Fundamental Theorem of Linear Programming:

- Given that an optimal solution to a linear programming problem exists, it must occur at a vertex of the feasible set.
- If the optimal solution occurs at two adjacent vertices of the feasible set, then the linear programming problem has infinitely many solutions. Any point on the line segment joining the two vertices is also a solution.

A graphical method for solving linear programming problems is outlined below.

Solving Linear Programming Problems – The Graphical Method

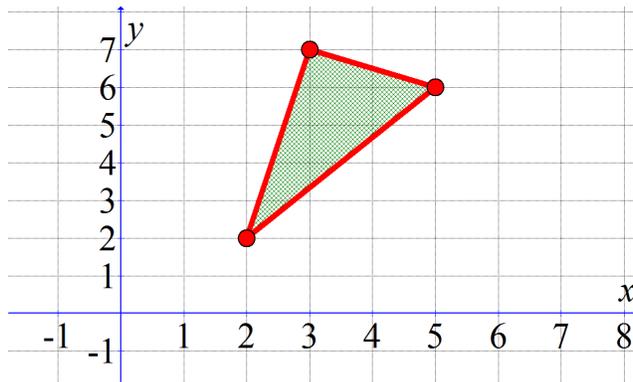
1. Graph the system of constraints. This will give the feasible set.
2. Find each vertex (corner point) of the feasible set.
3. Substitute each vertex into the objective function to determine which vertex optimizes the objective function.
4. State the solution to the problem.

An **unbounded set** is a set that has no bound and continues indefinitely. A linear programming problem with an unbounded set may or may not have an optimal solution, but if there is an optimal solution, it occurs at a corner point.

A **bounded set** is a set that has a boundary around the feasible set. A linear programming problem with a bounded set always has an optimal solution. This means that a bounded set has a maximum value as well as a minimum value.

Example 1: Given the objective function $P = 10x - 3y$ and the following feasible set,

- A. Find the maximum value and the point where the maximum occurs.
- B. Find the minimum value and the point where the minimum occurs.



Solution: We can see from the diagram that the feasible set is bounded, so this problem will have an optimal solution for the maximum as well as for the minimum. The vertices (corner points) of the feasible set are $(2, 2)$, $(3, 7)$, and $(5, 6)$.

To find the optimal solutions at which the maximum and minimum occur, we substitute each corner point into the objective function, $P = 10x - 3y$.

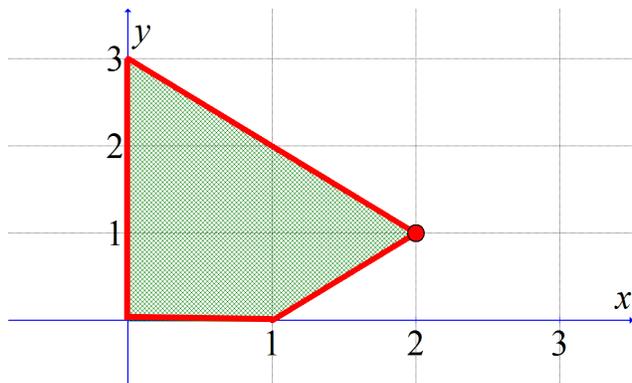
Vertex of Feasible Set	Value of $P = 10x - 3y$
(2, 2)	$P = 10(2) - 3(2) = 14$
(3, 7)	$P = 10(3) - 3(7) = 9$
(5, 6)	$P = 10(5) - 3(6) = 32$

We now look at our chart for the highest function value (the maximum) and the lowest function value (the minimum).

- A. The maximum value is 32 and it occurs at the point (5, 6).
- B. The minimum value is 9 and it occurs at the point (3, 7).

Example 2: Given the objective function $D = 5x + 8y$ and the following feasible set,

- A. Find the maximum value and the point where the maximum occurs.
- B. Find the minimum value and the point where the minimum occurs.



Solution: We can see from the diagram that the feasible set is bounded, so this problem will have an optimal solution for the maximum as well as for the minimum. The vertices (corner points) of the feasible set are (0, 0), (0, 3), (1, 0), and (2, 1). To find the optimal solutions at which the maximum and minimum occur, we substitute each corner point into the objective function, $D = 5x + 8y$.

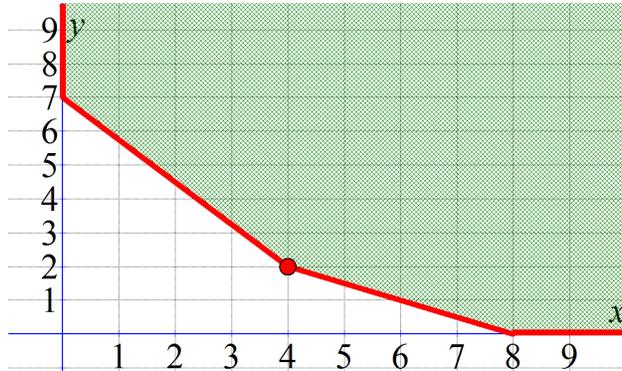
Vertex of Feasible Set	Value of $D = 5x + 8y$
(0, 0)	$D = 5(0) + 8(0) = 0$
(0, 3)	$D = 5(0) + 8(3) = 24$
(1, 0)	$D = 5(1) + 8(0) = 5$
(2, 1)	$D = 5(2) + 8(1) = 18$

We now look at our chart to find the maximum and the minimum of the objective function.

- A. The maximum value is 24 and it occurs at (0, 3).
- B. The minimum value is 0 and it occurs at (0, 0).

Example 3: Given the objective function $C = 12x + 4y$ and the following feasible set,

- A. Find the maximum value.
- B. Find the minimum value.



Solution: Notice that the feasible set is unbounded. This means that there may or may not be an optimal solution which results in a maximum or minimum function value. The vertices (corner points) of the feasible set are (0, 7), (4, 2), and (8, 0).

We substitute each corner point into the objective function, as shown in the chart below.

Vertex of Feasible Set	Value of $C = 12x + 4y$
(0, 7)	$C = 12(0) + 4(7) = 28$
(4, 2)	$C = 12(4) + 4(2) = 56$
(8, 0)	$C = 12(8) + 4(0) = 96$

- A. The maximum value appears to be 96, but remember that this set was unbounded. If we can find a point in the feasible set which yields a function value greater than 96, then this objective function has no maximum value.

Since the point (8, 0) yields the apparent maximum value of 96, let us choose some points in the feasible set near (8, 0) as test points.

For test point (9, 0): $C = 12x + 4y = 12(9) + 4(0) = 108$.

For test point (10, 2): $C = 12x + 4y = 12(10) + 4(2) = 128$.

(Although two examples are shown, we can stop looking as soon as we find one function value greater than 96.)

Since there are other points in the feasible set which yield an objective function value greater than 96, we can conclude that this function has no maximum.

- B. The minimum value appears to be 28, but since the feasible set is unbounded, this may or may not represent the minimum. Let us test some points in the feasible set near the suspected optimal solution of (0, 7):

For test point (0, 8): $C = 12x + 4y = 12(0) + 4(8) = 32$

For test point (1, 6): $C = 12x + 4y = 12(1) + 4(6) = 36$

For test point (1, 7): $C = 12x + 4y = 12(1) + 4(8) = 44$

The above is not a formal proof since we have only chosen a few points, but we could go on testing and would not find any points in the feasible set which yield a function value less than 28. The minimum value is 28.

Example 4: Use the graphical method to solve the following linear programming problem.

Maximize $R = 4x + 11y$

$$\text{subject to: } \begin{cases} x + y \leq 3 \\ 2x + y \leq 4 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

Solution: We need to graph the system of inequalities to produce the feasible set. We will start by rewriting each inequality as an equation, and then number the equation for each line.

$$x + y = 3 \quad (1)$$

$$2x + y = 4 \quad (2)$$

$$x = 0 \quad (3)$$

$$y = 0 \quad (4)$$

We want to graph each of the lines and determine the proper shading for their respective inequalities.

To graph Line (1), we will find its intercepts.

$$x + y = 3$$

$$x + 0 = 3$$

$$x = 3$$

$$x + y = 3$$

$$0 + y = 3$$

$$y = 3$$

The x - and y -intercepts for Line (1) are $(3, 0)$ and $(0, 3)$, respectively. Since the inequality $x + y \leq 3$ contains an equal sign, a solid line can be drawn through those two intercepts. We need to choose a test point to substitute into the original inequality to determine which half-plane to shade. We will choose the point $(0, 0)$:

$$x + y \leq 3$$

?

$$0 + 0 \leq 3$$

$$0 \leq 3$$

The point $(0, 0)$ satisfies the inequality, so we will shade the half-plane containing $(0, 0)$.

To graph Line (2), we will find its intercepts.

$$2x + y = 4$$

$$2x + 0 = 4$$

$$x = 2$$

$$2x + y = 4$$

$$2(0) + y = 4$$

$$y = 4$$

The x - and y -intercepts for Line (2) are $(2, 0)$ and $(0, 4)$, respectively. Since the inequality $2x + y \leq 4$ contains an equal sign, a solid line can be drawn through those two intercepts. We need to choose a test point to substitute into the original inequality to determine which half-plane to shade. We will choose the point $(0, 0)$:

$$2x + y \leq 4$$

$$\quad \quad ?$$

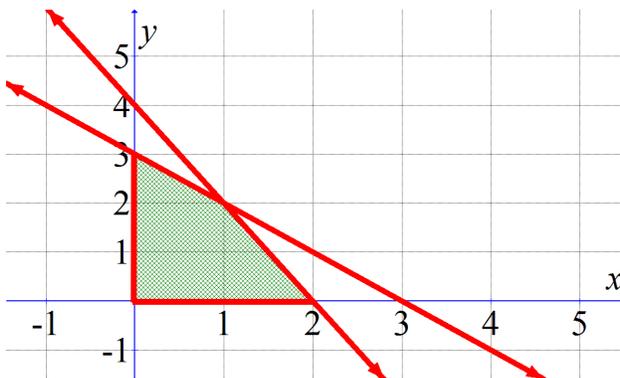
$$2(0) + 0 \leq 4$$

$$0 \leq 4$$

The point $(0, 0)$ satisfies the inequality, so we will shade the half-plane containing $(0, 0)$.

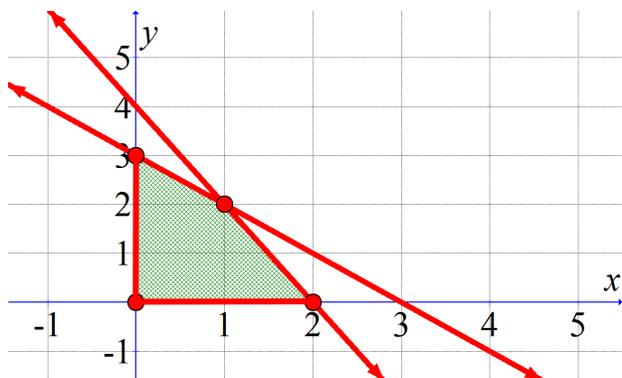
Lines (3) and (4) represent the y -axis and x -axis, respectively. The inequalities $x \geq 0$ and $y \geq 0$ together represent the first quadrant, so Quadrant I should be shaded.

The feasible set, shown below, is where all shaded regions intersect, along with the solid boundary of the shaded region.



We can see from the diagram that the feasible set is bounded, so this problem will have an optimal solution.

Next, we need to find the vertices (corner points) of the feasible set. By observing the graph we see that the vertices are $(0, 0)$, $(0, 3)$, $(1, 2)$, and $(2, 0)$.



Finally, we substitute each corner point into the objective function to determine the optimal solution.

Vertex of Feasible Set	Value of $R = 4x + 11y$
$(0, 0)$	$R = 4(0) + 11(0) = 0$
$(0, 3)$	$R = 4(0) + 11(3) = 33$
$(1, 2)$	$R = 4(1) + 11(2) = 26$
$(2, 0)$	$R = 4(2) + 11(0) = 8$

The maximum value is 33 and it occurs at $(0, 3)$.

Example 5: Use the graphical method to solve the following linear programming problem.

Minimize $S = 2x + 7y$

$$\text{subject to: } \begin{cases} 5x + y \geq 5 \\ x + 3y \geq 9 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

Solution: We need to graph the system of inequalities to produce the feasible set. We will start by rewriting each inequality as an equation, and then number the equation for each line.

$$5x + y = 5 \quad (1)$$

$$x + 3y = 9 \quad (2)$$

$$x = 0 \quad (3)$$

$$y = 0 \quad (4)$$

We want to graph each of the lines and determine the proper shading for their respective inequalities.

To graph Line (1), we will find its intercepts.

$$5x + y = 5$$

$$5x + 0 = 5$$

$$x = 1$$

$$5x + y = 5$$

$$5(0) + y = 5$$

$$y = 5$$

The x - and y -intercepts for Line (1) are $(1, 0)$ and $(0, 5)$, respectively. Since the inequality $5x + y \geq 5$ contains an equal sign, a solid line can be drawn through those two intercepts. We need to choose a test point to substitute into the original inequality to determine which half-plane to shade. We will choose the point $(0, 0)$:

$$5x + y \geq 5$$

$$5(0) + 0 \stackrel{?}{\geq} 5$$

$$0 \not\geq 5$$

The point $(0, 0)$ does not satisfy the inequality, so we will shade the half-plane that does not contain $(0, 0)$.

To graph Line (2), we will find its intercepts.

$$x + 3y = 9$$

$$x + 3(0) = 9$$

$$x = 9$$

$$x + 3y = 9$$

$$0 + 3y = 9$$

$$y = 3$$

The x - and y -intercepts for Line (2) are $(9, 0)$ and $(0, 3)$, respectively. Since the inequality $x + 3y \geq 9$ contains an equal sign, a solid line can be drawn through those two intercepts. We need to choose a test point to substitute into the original inequality to determine which half-plane to shade. We will choose the point $(0, 0)$:

$$x + 3y \geq 9$$

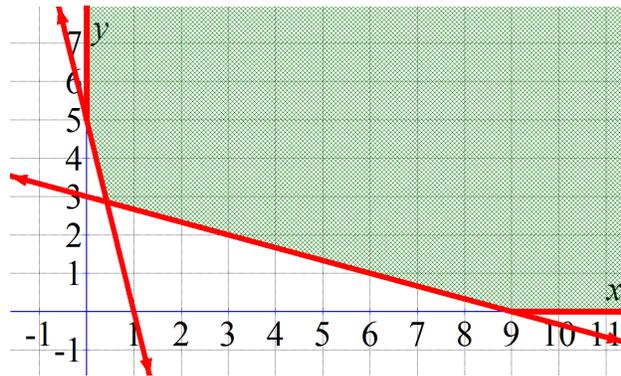
$$0 + 3(0) \stackrel{?}{\geq} 9$$

$$0 \not\geq 9$$

The point $(0, 0)$ does not satisfy the inequality, so we will shade the half-plane that does not contain $(0, 0)$.

Lines (3) and (4), represent the y -axis and x -axis, respectively. The inequalities $x \geq 0$ and $y \geq 0$ together represent the first quadrant, so Quadrant I should be shaded.

The feasible set, shown below, is where all shaded regions intersect, along with any solid boundaries of the shaded region.



We can see from the diagram that the feasible set is unbounded, so this problem may or may not have an optimal solution.

Next we need to find the vertices (corner points) of the feasible set. From examination of the graph along with previously found intercepts, we can see that the points $(0, 5)$ and $(9, 0)$ are corner points.

The point of intersection of Line (1) and Line (2) is not clear by simply looking at the graph, so we will find it algebraically by solving the following system of equations:

$$\begin{cases} 5x + y = 5 \\ x + 3y = 9 \end{cases}$$

We will choose to eliminate the variable y by the elimination method. The least common multiple of y and $3y$ is $3y$. We can keep the second equation the same so that it contains the term $3y$, and multiply the first equation by -3 so that it contains the term $-3y$.

Multiplying the first equation by -3 :

$$-15x - 3y = -15$$

We now add this resulting equation to the second equation:

$$\begin{array}{r} -15x - 3y = -15 \\ + \quad x + 3y = 9 \\ \hline -14x = -6 \\ x = \frac{3}{7} \end{array}$$

Next, we substitute $x = \frac{3}{7}$ into equation (1) or equation (2) to obtain the corresponding y-value.

$$\begin{aligned} 5x + y &= 5 \\ 5\left(\frac{3}{7}\right) + y &= 5 \\ y &= 5 - \frac{15}{7} = \frac{35}{7} - \frac{15}{7} = \frac{20}{7} \end{aligned}$$

The point of intersection of Line (1) and Line (2) is $\left(\frac{3}{7}, \frac{20}{7}\right)$, which gives us our final corner point.



The vertices of the feasible set are $(0, 5)$, $\left(\frac{3}{7}, \frac{20}{7}\right)$, and $(9, 0)$.

Next, we substitute each point into the objective function to determine the optimal solution.

Vertex of Feasible Set	Value of $S = 2x + 7y$
$(0, 5)$	$S = 2(0) + 7(5) = 35$
$\left(\frac{3}{7}, \frac{20}{7}\right)$	$S = 2\left(\frac{3}{7}\right) + 7\left(\frac{20}{7}\right) = \frac{146}{7} = 20\frac{6}{7}$
$(9, 0)$	$S = 2(9) + 7(0) = 18$

The minimum value appears to be 18, which occurs at $(9, 0)$. Since the feasible set is unbounded, this may or may not represent the minimum. Let us test some points in the feasible set near the suspected optimal solution of $(9, 0)$:

For test point $(8, 1)$: $S = 2x + 7y = 2(8) + 7(1) = 23$

For test point $(9, 1)$: $S = 2x + 7y = 2(9) + 7(1) = 25$

For test point $(10, 0)$: $S = 2x + 7y = 2(10) + 7(0) = 20$

The above is not a formal proof since we have only chosen a few points, but we could go on testing and would not find any points in the feasible set which yield a function value less than 18. The minimum value is 18, which occurs at $(9, 0)$.

Example 6: Use the graphical method to solve the following linear programming problem.

Maximize and minimize $Q = 3x + 5y$

$$\text{subject to: } \begin{cases} x + y \leq 6 \\ 2x + y \geq 8 \\ y \geq 0 \end{cases}$$

Solution: We need to graph the system of inequalities to produce the feasible set. We will start by rewriting each inequality as an equation, and then number the equation for each line.

$$x + y = 6 \quad (1)$$

$$2x + y = 8 \quad (2)$$

$$y = 0 \quad (3)$$

We want to graph each of the lines and determine the proper shading for their respective inequalities.

To graph Line (1), we will find its intercepts.

$$\begin{aligned}
 x + y &= 6 \\
 x + 0 &= 6 \\
 x &= 6
 \end{aligned}$$

$$\begin{aligned}
 x + y &= 6 \\
 0 + y &= 6 \\
 y &= 6
 \end{aligned}$$

The x - and y -intercepts for Line (1) are $(6, 0)$ and $(0, 6)$, respectively. Since the inequality $x + y \leq 6$ contains an equal sign, a solid line can be drawn through those two intercepts. We need to choose a test point to substitute into the original inequality to determine which half-plane to shade. We will choose the point $(0, 0)$:

$$\begin{aligned}
 x + y &\leq 6 \\
 \quad \quad ? \\
 0 + 0 &\leq 6 \\
 0 &\leq 6
 \end{aligned}$$

The point $(0, 0)$ satisfies the inequality, so we will shade the half-plane containing $(0, 0)$.

To graph Line (2), we will find its intercepts.

$$\begin{aligned}
 2x + y &= 8 \\
 2x + 0 &= 8 \\
 x &= 4
 \end{aligned}$$

$$\begin{aligned}
 2x + y &= 8 \\
 2(0) + y &= 8 \\
 y &= 8
 \end{aligned}$$

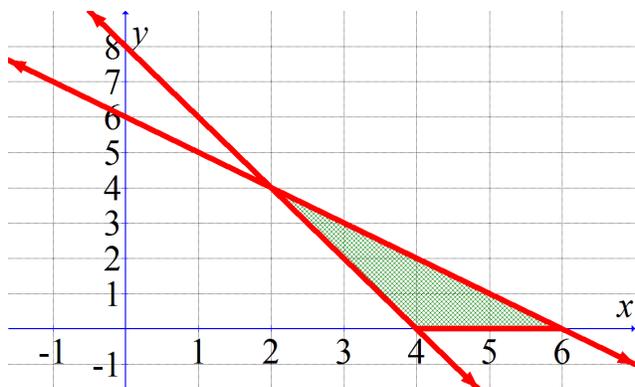
The x - and y -intercepts for Line (2) are $(4, 0)$ and $(0, 8)$, respectively. Since the inequality $2x + y \geq 8$ contains an equal sign, a solid line can be drawn through those two intercepts. We need to choose a test point to substitute into the original inequality to determine which half-plane to shade. We will choose the point $(0, 0)$:

$$\begin{aligned}
 2x + y &\geq 8 \\
 \quad \quad ? \\
 2(0) + 0 &\geq 8 \\
 0 &\geq 8
 \end{aligned}$$

The point $(0, 0)$ does not satisfy the inequality, so we will shade the half-plane that does not contain $(0, 0)$.

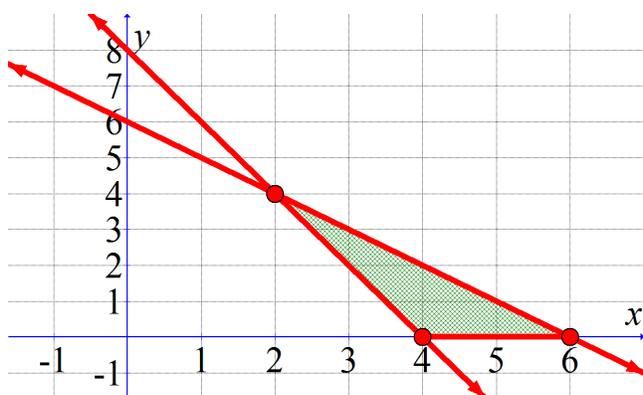
Line (3) is the x -axis. Since its associated inequality is $y \geq 0$, we should shade Quadrants I and II.

The feasible set, shown below, is where all shaded regions intersect, along with any solid boundaries of the shaded region.



We can see from the diagram that the feasible set is bounded, so this problem will have both a maximum and a minimum.

Next, we need to find the vertices (corner points) of the feasible set. By observing the graph we see that the vertices are $(4, 0)$, $(6, 0)$, and $(2, 4)$.



Finally, substitute each point into the objective function to determine the optimal solution.

Vertex of Feasible Set	Value of $U = 3x + 5y$
$(4, 0)$	$U = 2(4) + 5(0) = 8$
$(6, 0)$	$U = 2(6) + 5(0) = 12$
$(2, 4)$	$U = 2(2) + 5(4) = 24$

The maximum value is 24 and it occurs at $(2, 4)$.

The minimum value is 8 and it occurs at $(4, 0)$.

Example 7: Use the graphical method to solve the following linear programming problem.

Maximize and minimize $T = 5x + 6y$

$$\text{subject to: } \begin{cases} x + 2y \geq 8 \\ 6x + 4y \geq 36 \\ 0 \leq x \leq 7 \end{cases}$$

Solution: We need to graph the system of inequalities to produce the feasible set. We will start by rewriting each inequality as an equation (or in the case of the compound inequality, two equations), and then number the equation for each line.

$$x + 2y = 8 \quad (1)$$

$$6x + 4y = 36 \quad (2)$$

$$x = 0 \quad (3)$$

$$x = 7 \quad (4)$$

We want to graph each of the lines and determine the proper shading for their respective inequalities.

To graph Line (1), we will find its intercepts.

$$x + 2y = 8$$

$$x + 2(0) = 8$$

$$x = 8$$

$$x + 2y = 8$$

$$0 + 2y = 8$$

$$y = 4$$

The x - and y -intercepts for Line (1) are $(8, 0)$ and $(0, 4)$, respectively. Since the inequality $x + 2y \geq 8$ contains an equal sign, a solid line can be drawn through those two intercepts. We need to choose a test point to substitute into the original inequality to determine which half-plane to shade. We will choose the point $(0, 0)$:

$$x + 2y \geq 8$$

$$0 + 2(0) \stackrel{?}{\geq} 8$$

$$0 \not\geq 8$$

The point $(0, 0)$ does not satisfy the inequality, so we will shade the half-plane that does not contain $(0, 0)$.

To graph Line (2), we will find its intercepts.

$$6x + 4y = 36$$

$$6x + 4(0) = 36$$

$$x = 6$$

$$6x + 4y = 36$$

$$6(0) + 4y = 36$$

$$y = 9$$

The x - and y -intercepts for Line (2) are $(6, 0)$ and $(0, 9)$, respectively. Since the inequality $6x + 4y \geq 36$ contains an equal sign, a solid line can be drawn through those two intercepts. We need to choose a test point to substitute into the original inequality to determine which half-plane to shade. We will choose the point $(0, 0)$:

$$6x + 4y \geq 36$$

$$6(0) + 4(0) \geq 36$$

$$0 \geq 36$$

The point $(0, 0)$ does not satisfy the inequality, so we will shade the half-plane that does not contain $(0, 0)$.

Line (3) is $x = 0$, which represents the y -axis, and Line (4) is $x = 7$. Since $0 \leq x \leq 7$, we shade the area between these two solid vertical lines.

The feasible set, shown below, is where all shaded regions intersect, along with any solid boundaries of the shaded region.



We can see from the diagram that the feasible set is unbounded, so this problem may or may not have a maximum or a minimum.

Next we need to find the vertices (corner points) of the feasible set. From examination of the graph along with previously found intercepts, we can see that the point $(0, 9)$ is a corner point.

The other two corner points are not clear by simply looking at the graph, so we will find them algebraically by solving two systems of equations.

First, we solve the following system to find the intersection of Line (1) and Line (2):

$$\begin{cases} x + 2y = 8 \\ 6x + 4y = 36 \end{cases}$$

We will choose to eliminate the variable x by the elimination method. The least common multiple of x and $6x$ is $6x$. We can keep the second equation the same so that it contains the term $6x$, and multiply the first equation by -6 so that it contains the term $-6x$.

Multiplying the first equation by -6 :

$$-6x - 12y = -48$$

We now add this resulting equation to the second equation:

$$\begin{array}{r} -6x - 12y = -48 \\ + \quad 6x + 4y = 36 \\ \hline -8y = -12 \\ y = \frac{3}{2} \end{array}$$

Next, we substitute $y = \frac{3}{2}$ into equation (1) or equation (2) to obtain the corresponding x -value.

$$\begin{aligned} x + 2y &= 8 \\ x + 2\left(\frac{3}{2}\right) &= 8 \\ x &= 5 \end{aligned}$$

The point of intersection of Line (1) and Line (2) is $\left(5, \frac{3}{2}\right)$.

Next, we solve the following system to find the intersection of Line (1) and Line (4):

$$\begin{cases} x + 2y = 8 \\ x = 7 \end{cases}$$

We can use the substitution method, and substitute $x = 7$ into $x + 2y = 8$ to find the corresponding y -value.

$$x + 2y = 8$$

$$7 + 2y = 8$$

$$2y = 1$$

$$y = \frac{1}{2}$$

The point of intersection of Line (1) and Line (4) is $\left(7, \frac{1}{2}\right)$, which is our final corner point.



The vertices of the feasible set are $(0, 9)$, $\left(5, \frac{3}{2}\right)$, and $\left(7, \frac{1}{2}\right)$.

Next, we substitute each point into the objective function to determine the optimal solution.

Vertex of Feasible Set	Value of $T = 5x + 6y$
$(0, 9)$	$T = 5(0) + 6(9) = 54$
$\left(5, \frac{3}{2}\right)$	$T = 5(5) + 6\left(\frac{3}{2}\right) = 34$
$\left(7, \frac{1}{2}\right)$	$T = 5(7) + 6\left(\frac{1}{2}\right) = 38$

The minimum value appears to be 34, which occurs at $\left(5, \frac{3}{2}\right)$. Since the feasible set is unbounded, this may or may not represent the minimum. Let us test some points in the feasible set near the suspected optimal solution of $\left(5, \frac{3}{2}\right)$:

For test point $(5, 2)$: $T = 5x + 6y = 5(5) + 6(2) = 37$

For test point $(6, 2)$: $T = 5x + 6y = 5(6) + 6(2) = 42$

For test point $(5, 3)$: $T = 5x + 6y = 5(5) + 6(3) = 43$

Once again, the above is not a formal proof since we have only chosen a few points, but we could go on testing and would not find any points in the feasible set which yield a function value less than 34.

The minimum value is 18, which occurs at $\left(5, \frac{3}{2}\right)$.

The maximum value appears to be 54, which occurs at $(0, 9)$. Since the feasible set is unbounded, this may or may not represent the maximum. Let us test some points in the feasible set near the suspected optimal solution of $(0, 9)$:

For test point $(1, 9)$: $T = 5x + 6y = 5(1) + 6(9) = 5 + 54 = 59$

Notice that the function value of 59 is larger than our suspected maximum of 54. There is no need to test any other points. We can conclude that there is no maximum value in this unbounded feasible set.
