Math Handbook

of Formulas, Processes and Tricks

(www.mathguy.us)

Calculus

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Note to Students

This Calculus Handbook was developed primarily through work with a number of AP Calculus classes, so it contains what most students need to prepare for the AP Calculus Exam (AB or BC) or a first-year college Calculus course. In addition, a number of more advanced topics have been added to the handbook to whet the student's appetite for higher level study.

It is important to note that some of the tips and tricks noted in this handbook, while generating valid solutions, may not be acceptable to the College Board or to the student's instructor. The student should always check with their instructor to determine if a particular technique that they find useful is acceptable.

Why Make this Handbook?

One of my main purposes for writing this handbook is to encourage the student to wonder, to ask "what about … ?" or "what if … ?" I find that students are so busy today that they don't have the time, or don't take the time, to find the beauty and majesty that exists within Mathematics. And, it is there, just below the surface. So be curious and seek it out.

The answers to all of the questions below are inside this handbook, but are seldom taught.

- What is oscillating behavior and how does it affect a limit?
- Is there a generalized rule for the derivative of a product of multiple functions?
- What's the partial derivative shortcut to implicit differentiation?
- What are the hyperbolic functions and how do they relate to the trigonometric functions?
- When can I simplify a difficult definite integral by breaking it into its even and odd components?
- What is Vector Calculus?

Additionally, ask yourself:

- Why … ? Always ask "why?"
- Can I come up with a simpler method of doing things than I am being taught?
- What problems can I come up with to stump my friends?

Those who approach math in this manner will be tomorrow's leaders. Are you one of them?

Please feel free to contact me at mathguy.us@gmail.com if you have any comments.

Thank you and best wishes!

Earl

Cover art by Rebecca Williams,
Twitter handle: @jolteonkitty

Calculus Handbook
Table of Contents

Calculus Handbook
Table of Contents

Calculus Handbook
Table of Contents

Calculus Handbook
Table of Contents

Useful Websites

Mathguy.us – Developed specifically for math students from Middle School to College, based on the author's extensive experience in professional mathematics in a business setting and in math tutoring. Contains free downloadable handbooks, PC Apps, sample tests, and more.
www.mathguy.us

Wolfram Math World – A premier site for mathematics on the Web. This site contains definitions, explanations and examples for elementary and advanced math topics.
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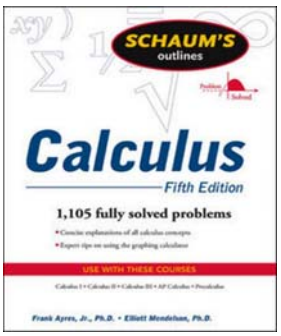
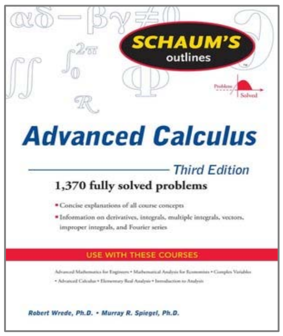
Calculus Handbook
Table of Contents

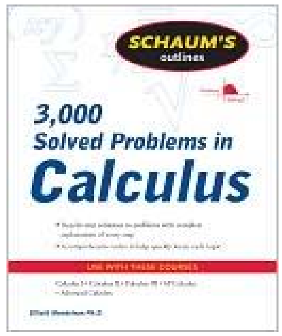
Schaum's Outlines

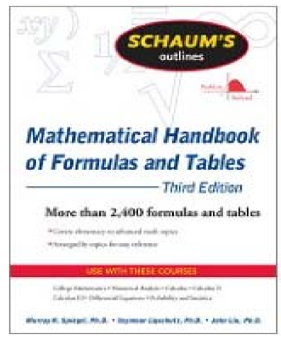
An important student resource for any high school math student is a Schaum's Outline. Each book in this series provides explanations of the various topics in the course and a substantial number of problems for the student to try. Many of the problems are worked out in the book, so the student can see how they can be solved.

Schaum's Outlines are available at Amazon.com, Barnes & Noble and other booksellers.

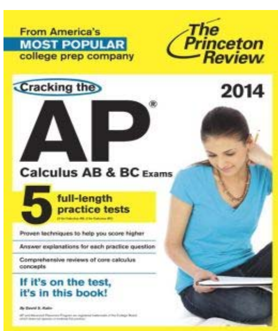


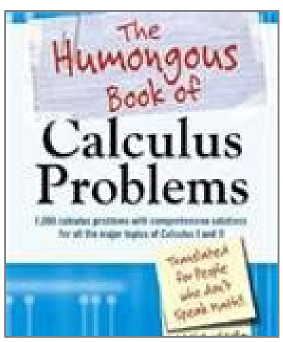


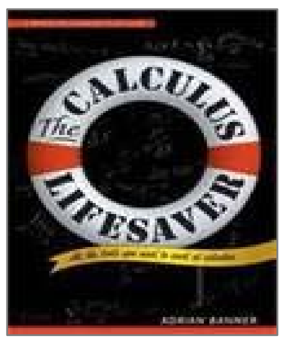


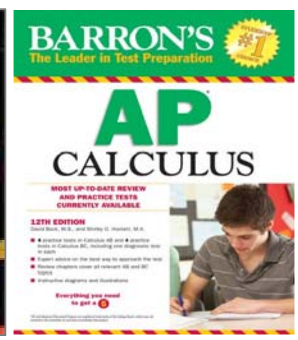


Other Useful Books









Functions

Definitions

- **Expression:** A meaningful arrangement of mathematical values, variables and operations.
- **Relation:** An expression that defines a connection between a set of inputs and a set of outputs. The set of inputs is called the **Domain** of the relation. The set of outputs is called the **Range** of the relation.
- **Function:** A relation in which each element in the domain corresponds to exactly one element in the range.
- **One-to-One Function:** A function in which each element in the range is produced by exactly one element in the domain.
- **Continuity:** A function, f, is continuous at $x = a$ iff:
  - $f(a)$ is defined,
  - $\lim_{x\to a} f(x)$ exists, and
  - $\lim_{x\to a} f(x) = f(a)$
  - If $x = a$ is an endpoint, then the limit need only exist from the left or the right.

Note: $\lim_{x\to a} f(x)$ exists if and only if: $\lim_{x\to a-} f(x) = \lim_{x\to a+} f(x)$.

Continuity Rules

If $f(x)$ and $g(x)$ are continuous functions at a point (x_0, y_0), and if c is a constant, then the following are also true at (x_0, y_0):

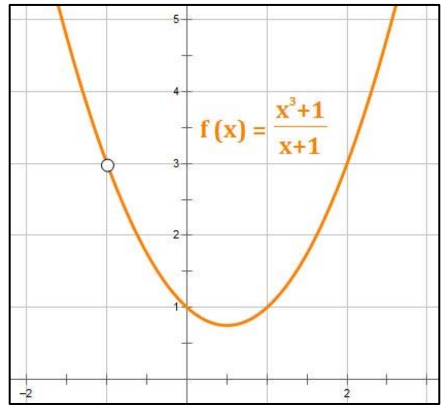
• $f(x) + g(x)$ is continuous.	Addition
• $f(x) - g(x)$ is continuous.	Subtraction
• $c \cdot f(x)$ is continuous.	Scalar Multiplication
• $f(x) \cdot g(x)$ is continuous.	Multiplication
• $\dfrac{f(x)}{g(x)}$ is continuous if $g(x_0) \neq 0$.	Division
• $f(x)^n$ is continuous if $f(x_0)^n$ exists.	Exponents
• $\sqrt[n]{f(x)}$ is continuous if $\sqrt[n]{f(x_0)}$ exists.	Roots

Note: All polynomial functions are continuous on the interval $(-\infty, +\infty)$.

Types of Discontinuities

A Discontinuity occurs at a location where the graph of a relation or function is not connected.

- Removable Discontinuity. A discontinuity that can be "repaired" by adding a single point to the graph. Typically, this will show up as a hole in a graph. In the function $f(x) = \frac{x^3+1}{x+1}$, a removable discontinuity exists at $x = -1$.



  Mathematically, a removable discontinuity is a point at which the limit of $f(x)$ at c exists but does not equal $f(c)$. That is,

$$\lim_{x \to c-} f(x) = \lim_{x \to c+} f(x) \neq f(c)$$

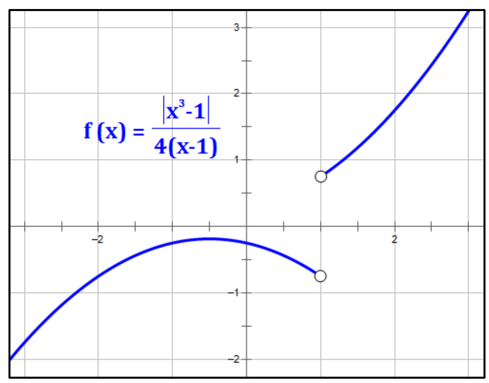
  Note: a removable discontinuity exists at $x = c$ whether or not $f(c)$ exists.

- Essential Discontinuity. A discontinuity that is not removable. Mathematically, an essential discontinuity is a point at which the limit of $f(x)$ at c does not exist. This includes:

  - Jump Discontinuity. A discontinuity at which the limit from the left does not equal the limit from the right. That is,

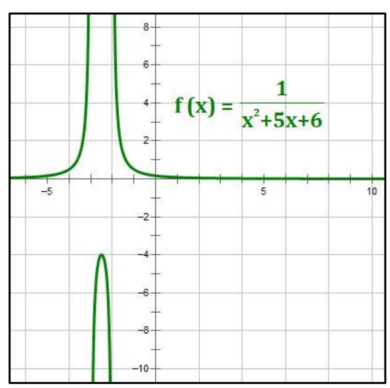
$$\lim_{x \to c-} f(x) \neq \lim_{x \to c+} f(x)$$

    In the function $f(x) = \frac{|x^3-1|}{4(x-1)}$, a jump discontinuity exists at $x = 1$.



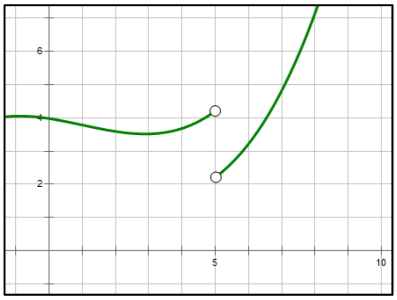
  - Infinite Discontinuity. These occur at vertical asymptotes.

    In the function $f(x) = \frac{1}{x^2+5x+6}$, infinite discontinuities exist at $x = \{-3, -2\}$.



Continuity Examples

Case 1



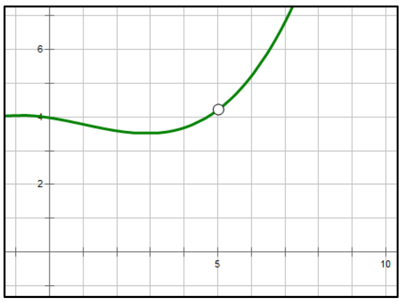
Jump Discontinuity

Not continuous

Limit does not exist

$f(5)$ may or may not exist (it does not exist in the graph shown)

Case 2



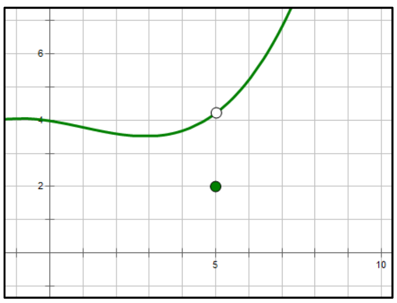
Removable Discontinuity

Not continuous

Limit exists

$f(5)$ does not exist

Case 3



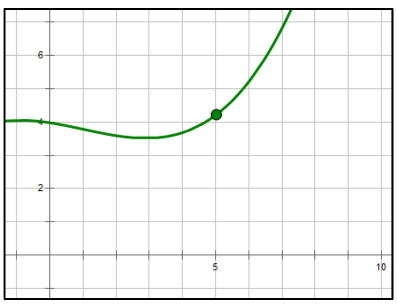
Removable Discontinuity

Not continuous

Limit exists

$f(5)$ exists but does not equal the limit

Case 4



No Discontinuity

Continuous

Limit exists

$f(5)$ exists and is equal the limit

Limits

Definitions

Formal Definition: Let $\boldsymbol{f}$ be a function defined on an open interval containing $\boldsymbol{a}$, except possibly at $\boldsymbol{x = a}$, and let $\boldsymbol{L}$ be a real number. Then, the statement:

$$\lim_{x \to a} f(x) = L$$

means that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that:

$$0 < |x - a| < \delta \quad \text{implies} \quad |f(x) - L| < \varepsilon.$$

Written using math symbols: $\forall\ \varepsilon > 0\ \ \exists\ \ \delta > 0\ \ \ni\ \ 0 < |x - a| < \delta\ \Rightarrow\ |f(x) - L| < \varepsilon.$

Informal Definition: The limit is the value $\boldsymbol{L}$ that a function approaches as the value of the input variable $\boldsymbol{x}$ approaches the desired value $\boldsymbol{a}$.

Limits may exist approaching $x = a$ from either the left $\left(\lim_{\boldsymbol{x \to a^-}} f(x)\right)$ or the right $\left(\lim_{\boldsymbol{x \to a^+}} f(x)\right)$. If the limits from the left and right are the same (e.g., they are both equal to L), then the limit exists at $x = a$ and we say $\lim_{x \to a} f(x) = L$.

Limit Rules

Assuming that each of the requisite limits exist, the following rules apply:

- $\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$ Addition of Limits
- $\lim_{x \to a}[f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$ Subtraction of Limits
- $\lim_{x \to a}[c \cdot f(x)] = c \cdot \lim_{x \to a} f(x)$ Scalar Multiplication
- $\lim_{x \to a}[f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$ Multiplication of Limits
- $\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ Division of Limits
- $\lim_{x \to a} f(x)^n = \left[\lim_{x \to a} f(x)\right]^n$ Powers
- $\lim_{x \to a}\left[\sqrt[n]{f(x)}\right] = \sqrt[n]{\lim_{x \to a} f(x)}$ Roots

Also, assuming that each of the requisite limits exists, the typical properties of addition and multiplication (e.g., commutative property, associative property, distributive property, inverse property, etc.) apply to limits.

Techniques for Finding Limits

Substitution

The easiest method, when it works, for determining a limit is **substitution.** Using this method, simply substitute the value of x into the limit expression to see if it can be calculated directly.

Example 1.1:

$$\lim_{x\to 3}\left(\frac{x+2}{x-2}\right)=\frac{3+2}{3-2}=\mathbf{5}$$

Simplification

When substitution fails, other methods must be considered. With rational functions (and some others), **simplification** may produce a satisfactory solution.

Example 1.2:

$$\lim_{x\to 5}\left(\frac{x^2-25}{x-5}\right)=\lim_{x\to 5}\left(\frac{(x+5)(x-5)}{(x-5)}\right)=x+5=\mathbf{10}$$

Rationalization

Rationalizing a portion of the limit expression is often useful in situations where a limit is **indeterminate**. In the example below the limit expression has the indeterminate form $(-\infty+\infty)$. Other indeterminate forms are discussed later in this chapter.

Example 1.3:

$$\lim_{x\to -\infty}\left(x+\sqrt{x^2-8x}\right)$$

First, notice that this limit is taken to $-\infty$, which can often cause confusion. So, let's modify it so that we are taking the limit to $+\infty$. We do this using the substitution $x=-y$.

$$\lim_{x\to -\infty}\left(x+\sqrt{x^2-8x}\right)=\lim_{y\to +\infty}\left(-y+\sqrt{y^2+8y}\right)$$

Next, let's rationalize the expression in the limit by multiplying by a name for one, using its conjugate.

(cont'd)

$$\lim_{y\to+\infty}\left(-y+\sqrt{y^2+8y}\right)=\lim_{y\to+\infty}\left(\frac{-y+\sqrt{y^2+8y}}{1}\cdot\frac{y+\sqrt{y^2+8y}}{y+\sqrt{y^2+8y}}\right)$$

$$=\lim_{y\to+\infty}\left(\frac{-y^2+y^2+8y}{y+\sqrt{y^2+8y}}\right)=\lim_{y\to+\infty}\left(\frac{8y}{y+\sqrt{y^2+8y}}\right)$$

$$=\lim_{y\to+\infty}\left(\frac{8y}{y+\sqrt{y^2+8y}}\div\frac{y}{y}\right)=\lim_{y\to+\infty}\left(\frac{8}{1+\sqrt{1+\frac{8}{y}}}\right)=\frac{8}{1+\sqrt{1}}=4$$

L'Hospital's Rule

If f and g are differentiable functions and $g'(x)\neq 0$ near a and if:

$$\lim_{x\to a}f(x)=0 \text{ and } \lim_{x\to a}g(x)=0 \qquad \textbf{OR} \qquad \lim_{x\to a}f(x)=\pm\infty \text{ and } \lim_{x\to a}g(x)=\pm\infty$$

$$\text{Then,} \qquad \boldsymbol{\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}}$$

Note: L'Hospital's rule can be repeated as many times as necessary *as long as the result of each step is an indeterminate form*. If a step produces a form that is not indeterminate, the limit should be calculated at that point.

Example 1.4:

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\frac{\mathrm{d}}{\mathrm{dx}}\sin x}{\frac{d}{dx}x}=\lim_{x\to 0}\frac{\cos x}{1}=\frac{1}{1}=\mathbf{1}$$

Example 1.5:

$$\lim_{x\to 0}\frac{x}{e^{3x}-1}=\lim_{x\to 0}\frac{\frac{\mathrm{d}}{\mathrm{dx}}x}{\frac{\mathrm{d}}{\mathrm{dx}}(e^{3x}-1)}=\frac{1}{3e^{3x}}=\frac{1}{3\cdot 1}=\mathbf{\frac{1}{3}}$$

Example 1.6: (involving successive applications of L'Hospital's Rule)

$$\lim_{x\to\infty}\frac{3x^3+2x+1}{4x^3-5x^2-2}=\lim_{x\to\infty}\frac{9x^2+2}{12x^2-10x}=\lim_{x\to\infty}\frac{18x}{24x-10}=\lim_{x\to\infty}\frac{18}{24}=\mathbf{\frac{3}{4}}$$

Indeterminate Forms of Limits

The following table presents some types of indeterminate forms that may be encountered and suggested methods for evaluating limits in those forms.

Form	Steps to Determine the Limit
$\frac{0}{0}$ or $\frac{\infty}{\infty}$	Use L'Hospital's Rule
$0 \cdot \infty$ $\infty - \infty$	For either of these forms: 1. Convert to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ 2. Use L'Hospital's Rule
0^0 ∞^0 1^∞	For any of these forms: 1. Take ln of the term or write the term in exponential form * 2. Convert to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ 3. Use L'Hospital's Rule

* For $y = [f(x)]^{g(x)}$, convert to: $\ln y = g(x) \cdot \ln f(x)$ or $y = e^{g(x) \cdot \ln f(x)}$

Example 1.7: **Form** $\mathbf{0 \cdot \infty}$

L'Hospital's Rule

$$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty} \left(\frac{x}{e^{-x}}\right) = \lim_{x\to-\infty} \left(\frac{1}{-e^{-x}}\right) = \lim_{x\to-\infty} -e^x = 0$$

Example 1.8: **Form** $\infty - \infty$

$$\lim_{x\to(\pi/2)^-} (\sec x - \tan x) = \lim_{x\to(\pi/2)^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$$

L'Hospital's Rule

$$= \lim_{x\to(\pi/2)^-} \frac{(1-\sin x)}{\cos x} = \lim_{x\to(\pi/2)^-} \frac{-\cos x}{-\sin x} = 0$$

Example 1.9: Form 0^0

$$\lim_{x \to 0^+} x^x \qquad \text{let:} \quad y = \lim_{x \to 0^+} x^x$$

L'Hospital's Rule

$$\ln y = \lim_{x \to 0^+} x \ \ln x = \lim_{x \to 0^+} \left(\frac{\ln x}{x^{-1}} \right) = \lim_{x \to 0^+} \left(\frac{x^{-1}}{-x^{-2}} \right)$$

$$= \lim_{x \to 0^+} (-x) = 0$$

$$\text{Then, since } \ \ln y = 0, \ \ \text{we get } \ y = e^0 = \mathbf{1}$$

Example 1.10: Form ∞^0

$$\lim_{x \to \infty} x^{1/x} \qquad \text{let:} \quad y = \lim_{x \to \infty} x^{1/x}$$

L'Hospital's Rule

$$\ln y = \lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \left(\frac{x^{-1}}{1} \right) = \lim_{x \to \infty} \left(\frac{1}{x} \right) = 0$$

$$\text{Then, since } \ \ln y = 0, \ \ \text{we get } \ y = e^0 = \mathbf{1}$$

Example 1.11: Form 1^∞

$$\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x} \qquad \text{let:} \quad y = \lim_{x \to 0^+} (1 + \sin 4x)^{\cot x}$$

$$\ln y = \lim_{x \to 0^+} [(\cot x) \cdot \ln(1 + \sin 4x)] = \lim_{x \to 0^+} \frac{\ln(1 + \sin 4x)}{\tan x}$$

L'Hospital's Rule

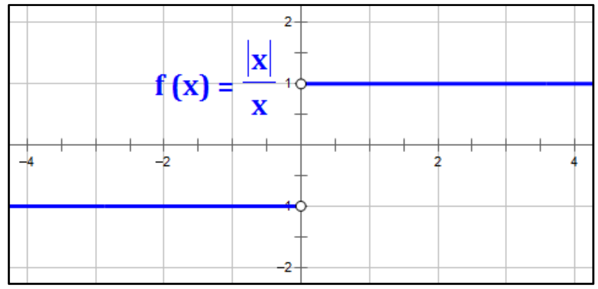
$$\ln y = \lim_{x \to 0^+} \frac{\left(\frac{4 \cos 4x}{1 + \sin 4x} \right)}{\sec^2 x} = \frac{\left(\frac{4 \cdot 1}{1 + 0} \right)}{1^2} = 4$$

$$\text{Then, since } \ \ln y = 4, \ \ \text{we get } \ y = e^4$$

When Limits Fail to Exist

There are several circumstances when limits fail to exist:

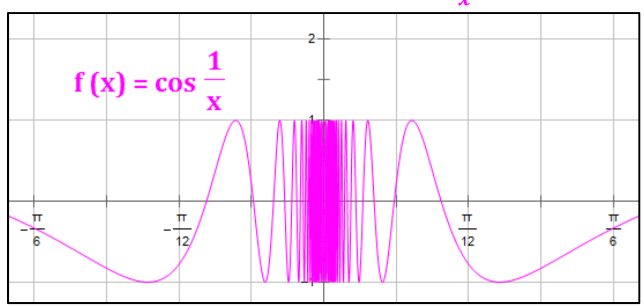
- When taken separately, limits from the left and right are different. This generally occurs at a jump discontinuity.

  In the graph of $(\boldsymbol{x}) = \dfrac{|\boldsymbol{x}|}{\boldsymbol{x}}$, a jump discontinuity exists at $x = 0$, so

  $$\lim_{x \to 0} \frac{|x|}{x} \quad \text{does not exist.}$$

- Oscillating behavior at the limit point. Consider the function $\boldsymbol{f(x) = \cos\frac{1}{x}}$, as $x \to 0$. In any neighborhood δ around $x = 0$, the value of the function varies from -1 to $+1$. Therefore,

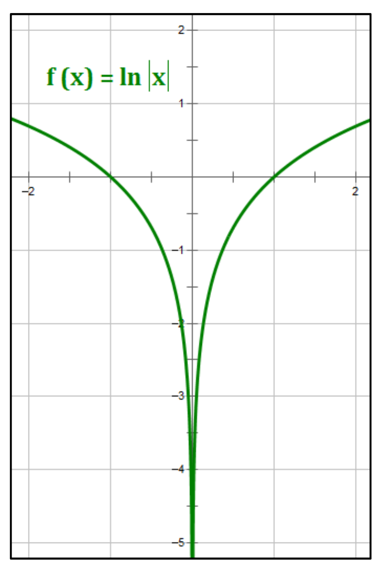
  

  $$\lim_{x \to 0} \left(\cos\frac{1}{x}\right) \quad \text{does not exist.}$$

  This function is also discontinuous at $x = 0$, though it is difficult to see this on the graph.

- Unbounded behavior at the limit point. Typically, this will happen at a vertical asymptote.

  In the graph of $\boldsymbol{f(x) = \ln|x|}$, an infinite discontinuity exists at $x = 0$ because the logarithms of positive real numbers that approach zero become large negative numbers without bound. Therefore,

  $$\lim_{x \to 0} \ln|x| \quad \text{does not exist.}$$

  Note: in this case, we may write: $\lim\limits_{x \to 0} \ln|x| = -\infty$

Basic Rules of Differentiation

Definition of a Derivative

$$\frac{d}{dx}\,f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \qquad\qquad \frac{d}{dx}\,f(x) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

Note: In these rules, c is a constant, and u and v are functions differentiable in x.

Basic Derivative Rules

$$\frac{d}{dx}(c) = 0 \qquad\qquad \frac{d}{dx}(c\cdot u) = c\cdot\frac{d}{dx}(u)$$

$$\frac{d}{dx}(u+v) = \frac{d}{dx}(u) + \frac{d}{dx}(v) \qquad\qquad \frac{d}{dx}(u-v) = \frac{d}{dx}(u) - \frac{d}{dx}(v)$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \qquad\qquad \frac{dy}{dx} = \frac{\frac{dy}{du}}{\frac{dx}{du}}$$

The Product, Quotient and Chain Rules are shown in Leibnitz, Lagrange, and differential forms.

Product Rule (two terms)

$$\frac{d}{dx}\,[\,f(x)\cdot g(x)\,] = f(x)\cdot\frac{d}{dx}[\,g(x)\,] + g(x)\cdot\frac{d}{dx}[\,f(x)\,]$$

$$(f\cdot g)' = f\cdot g' + g\cdot f'$$

$$d\,(uv) = u\,dv + v\,du$$

Product Rule (three terms)

$$\frac{d}{dx}\,[\,f(x)\cdot g(x)\,] = \frac{d}{dx}[\,f(x)\,]\cdot g(x)\cdot h(x) + f(x)\cdot\frac{d}{dx}[\,g(x)\,]\cdot h(x) + f(x)\cdot g(x)\cdot\frac{d}{dx}[\,h(x)\,]$$

$$(f\cdot g\cdot h)' = (f'\cdot g\cdot h) + (f\cdot g'\cdot h) + (f\cdot g\cdot h')$$

$$d\,(uvw) = vw\,du + uw\,dv + uv\,dw$$

Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot \frac{d}{dx}[f(x)] - f(x)\cdot\frac{d}{dx}[\,g(x)\,]}{[g(x)]^2}$$

$$\left(\frac{f}{g}\right)' = \frac{g\cdot f' - f\cdot g'}{g^2}$$

$$d\,\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$$

Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$h'(x) = f'\big(g(x)\big)\;\cdot\;g'(x),$ where: $h = f\circ g$

$$dy = \frac{dy}{du}\;\cdot\;du$$

Power Rule

$$\frac{d}{dx}(x^n) = n\cdot x^{n-1}$$

$$\frac{d}{dx}(u^n) = n\cdot u^{n-1}\frac{du}{dx}$$

Exponential and Logarithmic Functions $(a > 0, a \neq 1)$

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}e^u = e^u\cdot\frac{du}{dx}$$

$$\frac{d}{dx}a^x = a^x\cdot\ln a$$

$$\frac{d}{dx}a^u = a^u\cdot\ln a\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\log_a x = \frac{1}{x\,\ln a}$$

$$\frac{d}{dx}\log_a u = \frac{1}{u\,\ln a}\cdot\frac{du}{dx}$$

Derivatives of Special Functions
Trigonometric and Inverse Trigonometric Functions

Trigonometric Functions

$\frac{d}{dx}\sin x = \cos x$	$\frac{d}{dx}\sin u = \cos u \cdot \frac{du}{dx}$
$\frac{d}{dx}\cos x = -\sin x$	$\frac{d}{dx}\cos u = -\sin u \cdot \frac{du}{dx}$
$\frac{d}{dx}\tan x = \sec^2 x$	$\frac{d}{dx}\tan u = \sec^2 u \cdot \frac{du}{dx}$
$\frac{d}{dx}\cot x = -\csc^2 x$	$\frac{d}{dx}\cot u = -\csc^2 u \cdot \frac{du}{dx}$
$\frac{d}{dx}\sec x = \sec x \tan x$	$\frac{d}{dx}\sec u = \sec u \tan u \cdot \frac{du}{dx}$
$\frac{d}{dx}\csc x = -\csc x \cot x$	$\frac{d}{dx}\csc u = -\csc u \cot u \cdot \frac{du}{dx}$

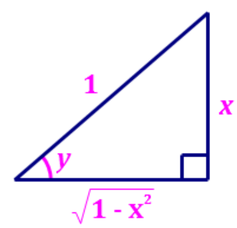
Inverse Trigonometric Functions (Basic Formulas)

$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$	**Angle in Q I or Q IV**
$\frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\cos^{-1} u = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$	**Angle in Q I or Q II**
$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$	$\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2} \cdot \frac{du}{dx}$	**Angle in Q I or Q IV**
$\frac{d}{dx}\cot^{-1} x = \frac{-1}{1+x^2}$	$\frac{d}{dx}\cot^{-1} u = \frac{-1}{1+u^2} \cdot \frac{du}{dx}$	**Angle in Q I or Q II**
$\frac{d}{dx}\sec^{-1} x = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{d}{dx}\sec^{-1} u = \frac{1}{\lvert u\rvert\sqrt{u^2-1}} \cdot \frac{du}{dx}$	**Angle in Q I or Q II**
$\frac{d}{dx}\csc^{-1} x = \frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{d}{dx}\csc^{-1} u = \frac{-1}{\lvert u\rvert\sqrt{u^2-1}} \cdot \frac{du}{dx}$	**Angle in Q I or Q IV**

Development of Basic Inverse Trig Derivatives

Inverse Sine

If $y = \sin^{-1} x,$ then $x = \sin y.$ Take the derivative of both sides of this equation, and consider the result in conjunction with the triangle at right.



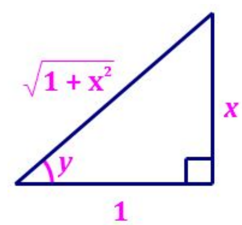
$$\sin y = x$$

$$\cos y \, \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$$

Inverse Tangent

If $y = \tan^{-1} x,$ then $x = \tan y.$ Take the derivative of both sides of this equation, and consider the result in conjunction with the triangle at right.



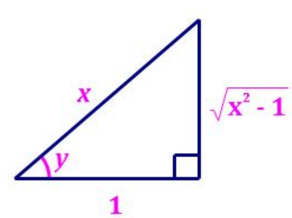
$$\tan y = x$$

$$\sec^2 y \, \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y = \left(\frac{1}{\sqrt{1 + x^2}}\right)^2 = \frac{1}{1 + x^2}$$

Inverse Secant

If $y = \sec^{-1} x,$ then $x = \sec y.$ Take the derivative of both sides of this equation, and consider the result in conjunction with the triangle at right.



$$\sec y = x$$

$$\sec y \tan y \, \frac{dy}{dx} = 1$$

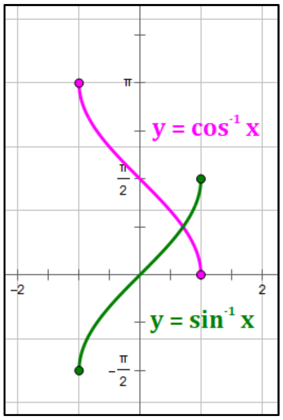
$$\frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{\cos^2 y}{\sin y} = \left(\frac{1}{x}\right)^2 \div \frac{\sqrt{x^2 - 1}}{|x|} = \frac{|x|}{x^2\sqrt{x^2 - 1}} = \frac{1}{|x|\sqrt{x^2 - 1}}$$

Note the use of the absolute value sign in this derivative. This occurs because the sec^{-1} *function is defined only in quadrants 1 and 2, and the sine function is always positive in these two quadrants. The student may verify that the slope of the* sec^{-1} *function is always positive.*

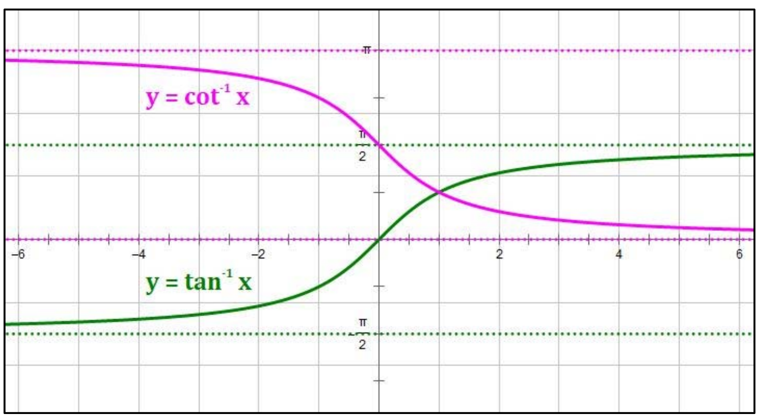
Graphs of Inverse Trig Functions

Graphs of the Inverse Trigonometric (IT) Functions over their principal ranges are provided below. Asymptotes are shown as dotted lines.

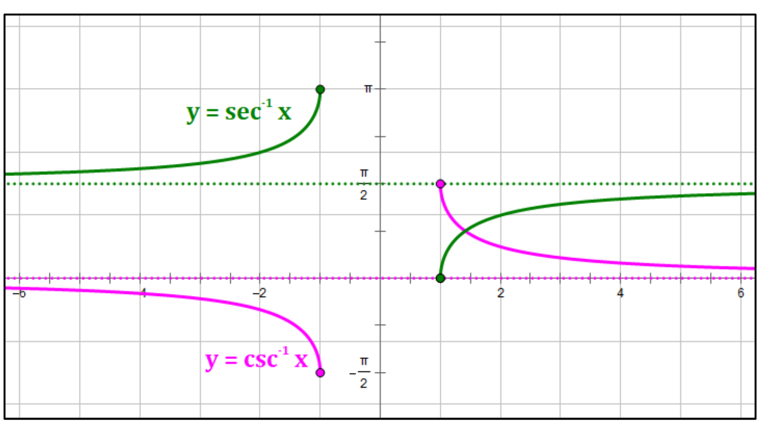
Notice the following about these graphs:



- The graphs of $\sin^{-1} x, \tan^{-1} x, \sec^{-1} x$ have positive slopes over their entire domains. So, their derivatives are always positive.
- The graphs of $\cos^{-1} x, \cot^{-1} x, \csc^{-1} x$ have negative slopes over their entire domains. So, their derivatives are always negative.
- Each IT function has a principal range of length π radians, i.e., two quadrants. In one of these quadrants, the corresponding trigonometric function value is negative, and in the other it is positive. For example, $\cos^{-1} x$ has range $[0, \pi]$, Quadrants I and II. In Quadrant I, $\cos x$ is positive and in Quadrant II, $\cos x$ is negative.



- At each x-value, cofunction pairs (e.g., $\sin^{-1} x$ and $\cos^{-1} x$) have slopes with opposite values, i.e., the same absolute value but one slope is positive while the other slope is negative.
- Cofunction pairs (e.g., $\sin^{-1} x$ and $\cos^{-1} x$) are reflections of each other over the horizontal line that contains their intersection.



- There is not universal agreement on the principal range of $\cot^{-1} x$. Some sources, including the TI nSpire and a number of Calculus textbooks, set the range to $(0, \pi)$, as shown on this page. Others, including Wolfram MathWorld and the US National Institute of Standards and Technology, set the range to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Generalized Inverse Trig Derivatives

Derivatives

Note that "a" is defined to be positive in these formulas in order to meet the domain restrictions of inverse Trigonometric functions.

$\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right)=\frac{1}{\sqrt{a^2-x^2}}$	$\frac{d}{dx}\sin^{-1}\left(\frac{u}{a}\right)=\frac{1}{\sqrt{a^2-u^2}}\cdot\frac{du}{dx}$	**Angle in Q I or Q IV**
$\frac{d}{dx}\cos^{-1}\left(\frac{x}{a}\right)=\frac{-1}{\sqrt{a^2-x^2}}$	$\frac{d}{dx}\cos^{-1}\left(\frac{u}{a}\right)=\frac{-1}{\sqrt{a^2-u^2}}\cdot\frac{du}{dx}$	**Angle in Q I or Q II**
$\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right)=\frac{a}{a^2+x^2}$	$\frac{d}{dx}\tan^{-1}\left(\frac{u}{a}\right)=\frac{a}{a^2+u^2}\cdot\frac{du}{dx}$	**Angle in Q I or Q IV**
$\frac{d}{dx}\cot^{-1}\left(\frac{x}{a}\right)=\frac{-a}{a^2+x^2}$	$\frac{d}{dx}\cot^{-1}\left(\frac{u}{a}\right)=\frac{-a}{a^2+u^2}\cdot\frac{du}{dx}$	**Angle in Q I or Q II**
$\frac{d}{dx}\sec^{-1}\left(\frac{x}{a}\right)=\frac{a}{\lvert x\rvert\sqrt{x^2-a^2}}$	$\frac{d}{dx}\sec^{-1}\left(\frac{u}{a}\right)=\frac{a}{\lvert u\rvert\sqrt{u^2-a^2}}\cdot\frac{du}{dx}$	**Angle in Q I or Q II**
$\frac{d}{dx}\csc^{-1}\left(\frac{x}{a}\right)=\frac{-a}{\lvert x\rvert\sqrt{x^2-a^2}}$	$\frac{d}{dx}\csc^{-1}\left(\frac{u}{a}\right)=\frac{-a}{\lvert u\rvert\sqrt{u^2-a^2}}\cdot\frac{du}{dx}$	**Angle in Q I or Q IV**

Sample Developments of Generalized Formulas from Basic Formulas

$$\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right)=\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\cdot\left(\frac{1}{a}\right)=\frac{1}{\sqrt{1-\frac{x^2}{a^2}}\cdot\sqrt{a^2}}=\frac{1}{\sqrt{a^2-x^2}}$$

$$\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right)=\frac{1}{1+\left(\frac{x}{a}\right)^2}\cdot\left(\frac{1}{a}\right)=\frac{1}{a+\frac{x^2}{a}}=\frac{1}{\frac{a^2}{a}+\frac{x^2}{a}}=\frac{1}{\frac{a^2+x^2}{a}}=\frac{a}{a^2+x^2}$$

$$\frac{d}{dx}\sec^{-1}\left(\frac{x}{a}\right)=\frac{1}{\left|\frac{x}{a}\right|\sqrt{\left(\frac{x}{a}\right)^2-1}}\cdot\left(\frac{1}{a}\right)=\frac{1}{\frac{\lvert x\rvert}{a}\sqrt{\frac{x^2}{a^2}-\frac{a^2}{a^2}}\cdot\sqrt{a^2}}=\frac{a}{\lvert x\rvert\sqrt{x^2-a^2}}$$

Generalized Product Rule

Product Rule (three terms)

$$\frac{d}{dx}(uvw) = \frac{d}{dx}[u(vw)] = u\frac{d}{dx}(vw) + vw\frac{du}{dx}$$

$$= u \cdot \left(v\frac{dw}{dx} + w\frac{dv}{dx}\right) + vw\frac{du}{dx}$$

$$\boldsymbol{\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}}$$

Product Rule (four terms)

$$\frac{d}{dx}(uvwt) = \frac{d}{dx}[(uv)(wt)] = uv\frac{d}{dx}(wt) + wt\frac{d}{dx}(uv)$$

$$= uv \cdot \left(w\frac{dt}{dx} + t\frac{dw}{dx}\right) + wt \cdot \left(u\frac{dv}{dx} + v\frac{du}{dx}\right)$$

$$\boldsymbol{\frac{d}{dx}(uvwt) = uvw\frac{dt}{dx} + uvt\frac{dw}{dx} + uwt\frac{dv}{dx} + vwt\frac{du}{dx}}$$

Generalized Product Rule (*n* terms)

$$\boldsymbol{\frac{d}{dx}\left(\prod_{i=1}^{n} u_i\right) = \sum_{j=1}^{n}\left[\left(\prod_{i \neq j} u_j\right) \cdot \frac{d}{dx}(u_j)\right]}$$

In words:

1. **Take the derivative of each function in the product.**
2. **Multiply it by all of the other functions in the product.**
3. **Add all of the resulting terms.**

Example 2.1: Product Rule (six terms) – from Generalized Product Rule

$$\boldsymbol{\frac{d}{dx}(uvwrst) = uvwrs\frac{dt}{dx} + uvwrt\frac{ds}{dx} + uvwst\frac{dr}{dx} + uvrst\frac{dw}{dx} + uwrst\frac{dv}{dx} + vwrst\frac{du}{dx}}$$

Generalized Product Rule
Example

In words:

1. **Take the derivative of each function in the product.**
2. **Multiply it by all of the other functions in the product.**
3. **Add all of the resulting terms.**

Generalized Product Rule (n terms)

$$\frac{d}{dx}\left(\prod_{i=1}^{n} u_i\right) = \sum_{j=1}^{n}\left[\left(\prod_{i\neq j} u_j\right)\cdot\frac{d}{dx}(u_j)\right]$$

Example 2.2: Find the derivative of: $f(x) = (4x^2 + x) \cdot e^x \cdot \sin 3x \cdot \cos x^2$

Let: $u = (4x^2 + x)$

$v = e^x$

$w = \sin 3x$

$t = \cos x^2$

Then, build the derivative based on the four components of the function:

Original Function Term	Derivative of Original Function Term	Remaining Functions
$u = (4x^2 + x)$	$8x + 1$	$e^x \cdot \sin 3x \cdot \cos x^2$
$v = e^x$	e^x	$(4x^2 + x) \cdot \sin 3x \cdot \cos x^2$
$w = \sin 3x$	$3\cos 3x$	$(4x^2 + x) \cdot e^x \cdot \cos x^2$
$t = \cos x^2$	$-2x \sin x^2$	$(4x^2 + x) \cdot e^x \cdot \sin 3x$

The resulting derivative is:

$$f'(x) = (8x+1)(e^x \cdot \sin 3x \cdot \cos x^2) + (4x^2 + x) \cdot e^x \cdot \sin 3x \cdot \cos x^2 +$$
$$3 \cdot (4x^2 + x) \cdot e^x \cdot \cos 3x \cdot \cos x^2 - (8x^3 + 2x^2) \cdot e^x \cdot \sin 3x \cdot \sin x^2$$

Inverse Function Rule

The **Inverse Function Rule** states the following:

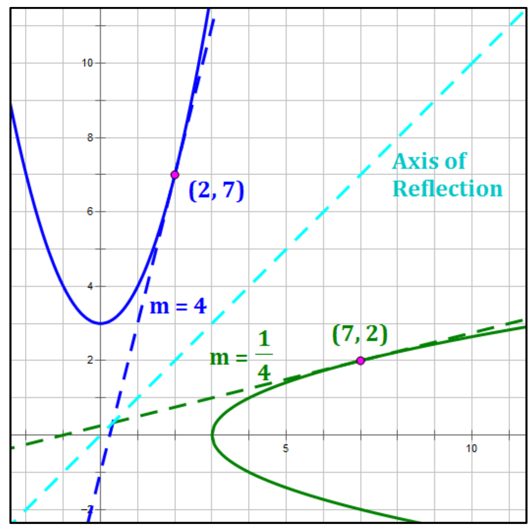
If $f(x)$ and $g(x)$ are inverse functions and $f'(g(x)) \neq 0$, then $g'(x) = \dfrac{1}{f'(g(x))}$

To understand what this means, it may be best to look at what it says graphically and create an **Inverse Function Diagram**.

Example 2.3: Let $f(x) = x^2 + 3$. Find the slope of $g(x) = f^{-1}(x)$ at the point $(7, 2)$.

To solve this, let's look at the graph of $\boldsymbol{f(x) = x^2 + 3}$ and its inverse $\boldsymbol{g(x) = \pm\sqrt{x - 3}}$.

The figure at right shows these two plots, along with the axis of reflection and the lines tangent to the two curves at the desired points.



Notice the following:

- $\boldsymbol{g(7) = 2}$, so $\boldsymbol{f(2) = 7}$
- $\boldsymbol{f'(x) = 2x}$, so $\boldsymbol{f'(2) = 4}$
- $\boldsymbol{g'(7) = \frac{1}{f'(2)} = \frac{1}{4}}$ (the answer)

An **Inverse Function Diagram (IFD)** organizes this information as follows:

IFD for Example 2.3	General IFD
$\boldsymbol{f(2) = 7 \quad \Leftrightarrow \quad g(7) = 2}$	$\boldsymbol{f(x_0) = y_0 \quad \Leftrightarrow \quad g(y_0) = x_0}$
$\downarrow$	$\downarrow$
$\boldsymbol{f'(2) = 4 \quad \xrightarrow{yields} \quad g'(7) = \frac{1}{4}}$	$\boldsymbol{f'(x_0) = m \quad \xrightarrow{yields} \quad g'(y_0) = \frac{1}{m}}$

Partial Differentiation

Partial differentiation is differentiation with respect to a single variable, with all other variables being treated as constants.

Example 2.4: Consider the function $f(x,y) = xy + 2x + 3y$.

Full derivative:	**Partial derivative:**	**Partial derivative:**
$\frac{d}{dx}(xy + 2x + 3y) =$	$\frac{\partial}{\partial x}(xy + 2x + 3y) =$	$\frac{\partial}{\partial y}(xy + 2x + 3y) =$
$x\frac{dy}{dx} + y + 2 + 3\frac{dy}{dx}$	$y + 2$	$x + 3$

Notice in the partial derivative panels above, that the "off-variable" is treated as a constant.

- In the left-hand panel, the derivative is taken in its normal manner, including using the product rule on the xy-term.
- In the middle panel, which takes the partial derivative with respect to x, y is considered to be the coefficient of x in the xy-term. In the same panel, the $3y$ term is considered to be a constant, so its partial derivative with respect to x is 0.
- In the right-hand panel, which takes the partial derivative with respect to y, x is considered to be the coefficient of y in the xy-term. In the same panel, the $2x$ term is considered to be a constant, so its partial derivative with respect to y is 0.

Partial derivatives provide measures of rates of change in the direction of the variable. So, for a 3-dimensional curve, $\frac{\partial z}{\partial x}$ provides the rate of change in the x-direction and $\frac{\partial z}{\partial y}$ provides the rate of change in the y-direction. Partial derivatives are especially useful in physics and engineering.

Example 2.5: Let $\boldsymbol{w = x^2 e^{3y} \ln z + e^{4x} \sin(y + z) - \cos(xyz)}$. Then,

$$\frac{\partial w}{\partial x} = 2xe^{3y} \ln z + 4e^{4x} \sin(y + z) + yz \sin(xyz)$$

$$\frac{\partial w}{\partial y} = 3x^2 e^{3y} \ln z + e^{4x} \cos(y + z) + xz \sin(xyz)$$

$$\frac{\partial w}{\partial z} = \frac{x^2 e^{3y}}{z} + e^{4x} \cos(y + z) + xy \sin(xyz)$$

Implicit Differentiation

Implicit differentiation is typically used when it is too difficult to differentiate a function directly. The entire expression is differentiated with respect to one of the variables in the expression, and algebra is used to simplify the expression for the desired derivative.

Example 2.6: Find $\frac{dy}{dx}$ for the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=36$.

We could begin by manipulating the equation to obtain a value for y:

$$\boldsymbol{y=\pm 3\sqrt{36-\frac{x^2}{4}}}.$$

However, this is a fairly ugly expression for y, and the process of developing $\frac{dy}{dx}$ is also ugly. It is many times easier to differentiate implicitly as follows:

1. Start with the given equation: $\frac{x^2}{4}+\frac{y^2}{9}=36$
2. Multiply both sides by 36 to get rid of the denominators: $9x^2+4y^2=1296$
3. Differentiate with respect to x: $18x+8y\cdot y'=0$
4. Subtract $18x$: $8y\cdot y'=-18x$
5. Divide by $8y$: $y'=\frac{-18x}{8y}=-\frac{9x}{4y}$
6. Sometimes you will want to substitute in the value of y to get the expression solely in terms of x: $y'=-\frac{9x}{4y}=-\frac{9x}{4\left(\pm 3\sqrt{36-\frac{x^2}{4}}\right)}$

$$\boldsymbol{y'=\pm\frac{3x}{4\sqrt{36-\frac{x^2}{4}}}} \qquad (x\neq\pm 12)$$

The result is still ugly and, in fact, it must be ugly. However, the algebra required to get the result may be cleaner and easier using implicit differentiation. In some cases, it is either extremely difficult or impossible to develop an expression for y in terms of x because the variables are so intertwined; see the example on the next page.

Implicit Differentiation (cont'd)

Example 2.7: Find $\frac{dy}{dx}$ for the equation: $x \cdot \sin y + y \cdot \cos x = 0$.

Manipulating this equation to find y as a function of x is out of the question. So, we use implicit differentiation as follows:

1. Start with the given equation: $\boldsymbol{x \cdot \sin y + y \cdot \cos x} = 0$

2. Differentiate with respect to x using the product rule and the chain rule:

$$\boldsymbol{x \cdot \frac{d}{dx}(\sin y) + \sin y \cdot \frac{d}{dx}(x) + y \cdot \frac{d}{dx}(\cos x) + \cos x \cdot \frac{d}{dx}(y)} = 0$$

3. Simplify:

$$\boldsymbol{x \cdot (\cos y) \cdot \frac{dy}{dx} + \sin y + y \cdot (-\sin x) + \cos x \cdot \frac{dy}{dx}} = 0$$

4. Combine like terms and simplify:

$$\boldsymbol{x \cdot (\cos y) \cdot \frac{dy}{dx} + \cos x \cdot \frac{dy}{dx} + \sin y + y \cdot (-\sin x)} = 0$$

$$[x \cdot (\cos y) + \cos x] \cdot \frac{dy}{dx} \;+\; [\sin y - y \cdot (\sin x)] = 0$$

$$[x \cdot (\cos y) + \cos x] \cdot \frac{dy}{dx} \;=\; [y \cdot (\sin x) - \sin y]$$

$$\boldsymbol{\frac{dy}{dx} = \frac{y \cdot (\sin x) - \sin y}{x \cdot (\cos y) + \cos x}} \qquad \text{(as long as: } x \cdot (\cos y) + \cos x \neq 0)$$

That's as good as we can do. Notice that the derivative is a function of both x and y. Even though we cannot develop an expression for y as a function of x, we can still calculate a derivative of the function in terms of x and y. Viva implicit differentiation!

Implicit Differentiation (cont'd)

Implicit Differentiation Using Partial Derivatives

Let $z = f(x, y)$. Then, the following formula is often a shortcut to calculating $\frac{dy}{dx}$.

$$\frac{dy}{dx} = -\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}}$$

Let's re-do the examples from the previous pages using the partial derivative method.

Example 2.8: Find $\frac{dy}{dx}$ for the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 36$.

Let: $z = \frac{x^2}{4} + \frac{y^2}{9} - 36$. Then,

$$\frac{\partial z}{\partial x} = \frac{2x}{4} \qquad \frac{\partial z}{\partial y} = \frac{2y}{9} \qquad \frac{dy}{dx} = -\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = -\frac{\frac{2x}{4}}{\frac{2y}{9}} = -\frac{9x}{4y}$$

Example 2.9: Find $\frac{dy}{dx}$ for the equation: $x \cdot \sin y + y \cdot \cos x = 0$.

Let: $z = x \sin y + y \cos x$. Then,

$$\frac{\partial z}{\partial x} = \sin y - y \sin x \qquad \frac{\partial z}{\partial y} = x \cos y + \cos x$$

$$\frac{dy}{dx} = -\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = -\frac{\sin y - y \sin x}{x \cos y + \cos x} = \frac{y \sin x - \sin y}{x \cos y + \cos x}$$

Contrast the work required here with the lengthy efforts required to calculate these results on the two prior pages.

So, implicit differentiation using partial derivatives can be fast and, because fewer steps are involved, improve accuracy. Just be careful how you handle each variable. This method is different and takes some getting used to.

Logarithmic Differentiation

Logarithmic differentiation is typically used when functions exist in both the base and the exponent of an exponential expression. Without this approach, the differentiation of the function would be much more difficult. The process involves several steps, as follows:

1. If possible, put the function in the form: $y = f(x)$
2. Take natural logarithms of both sides of the expression.
3. Take the derivatives of both sides of the expression.
4. Solve for $\frac{dy}{dx}$.

Example 2.10: Calculate the derivative of the general case $\boldsymbol{y = u^v}$, where u and v are functions of x, and are differentiable at x.

1. Original equation $\qquad \boldsymbol{y = u^v}$
2. Take natural logarithms of both sides $\qquad \boldsymbol{ln\, y = ln\, u^v}$
3. Simplify right side $\qquad \boldsymbol{ln\, y = v \cdot ln\, u}$
4. Take derivatives of both sides $\qquad \boldsymbol{\frac{1}{y} \cdot \frac{dy}{dx} = v \cdot \frac{d}{dx}(ln\, u) + (ln\, u)\frac{dv}{dx}}$
5. Apply Product Rule and Chain Rule to right side $\qquad \boldsymbol{\frac{1}{y} \cdot \frac{dy}{dx} = \left[v \cdot \frac{1}{u} \cdot \frac{du}{dx} + (ln\, u)\frac{dv}{dx}\right]}$
6. Multiply both sides by y $\qquad \boldsymbol{\frac{dy}{dx} = y \cdot \left[\frac{v}{u} \cdot \frac{du}{dx} + (ln\, u)\frac{dv}{dx}\right]}$
7. Substitute value of y $\qquad \boldsymbol{\frac{dy}{dx} = u^v \cdot \left[\frac{v}{u} \cdot \frac{du}{dx} + (\ln u)\frac{dv}{dx}\right]}$
8. Simplify $\qquad \boldsymbol{\frac{d}{dx}u^v = vu^{v-1} \cdot \frac{du}{dx} + u^v(\ln u)\frac{dv}{dx}}$

Maxima and Minima

Relative Extrema

Relative maxima and minima (also called relative extrema) may exist wherever the derivative of a function is either equal to zero or undefined. However, these conditions are not sufficient to establish that an extreme exists; we must also have a change in the direction of the curve, i.e., from increasing to decreasing or from decreasing to increasing.

Note: relative extrema cannot exist at the endpoints of a closed interval.

First Derivative Test

If

- a function, f, is continuous on the open interval (a, b), and
- c is a critical number $\in (a, b)$ (i.e., $f'(c)$ is either zero or does not exist),
- f is differentiable on the open interval (a, b), except possibly at *c,*

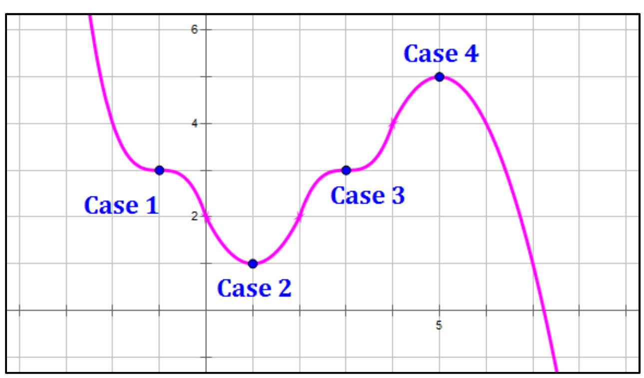
Then

- If $f'(x)$ changes from positive to negative at c, then $f(c)$ is a relative maximum.
- If $f'(x)$ changes from negative to positive at c, then $f(c)$ is a relative minimum.

The conclusions of this theorem are summarized in the table below:

<table>
<tr><th></th><th>First Derivative</th><th>Sign of $\frac{dy}{dx}$ left of $x = c$</th><th>Sign of $\frac{dy}{dx}$ right of $x = c$</th><th>Type of Extreme</th></tr>
<tr><td>Case 1</td><td rowspan="4">$\frac{dy}{dx} = 0$ or $\frac{dy}{dx}$ does not exist.</td><td>$-$</td><td>$-$</td><td>None</td></tr>
<tr><td>Case 2</td><td>$-$</td><td>$+$</td><td>Minimum</td></tr>
<tr><td>Case 3</td><td>$+$</td><td>$+$</td><td>None</td></tr>
<tr><td>Case 4</td><td>$+$</td><td>$-$</td><td>Maximum</td></tr>
</table>

Illustration of First Derivative Test for Cases 1 to 4:



Second Derivative Test

If

- a function, f, is continuous on the open interval (a, b), and
- $c \in (a, b)$, and
- $f'(c) = 0$ and $f''(c)$ exists,

Then

- If $f''(c) < 0$, then $f(c)$ is a relative maximum.
- If $f''(c) > 0$, then $f(c)$ is a relative minimum.

The conclusions of the theorem are summarized in the table below:

	First Derivative	**Second Derivative**	**Type of Extreme**
Case 1	$\frac{dy}{dx} = 0$ or $\frac{dy}{dx}$ does not exist.	$\frac{d^2y}{dx^2} < 0$	Maximum
Case 2		$\frac{d^2y}{dx^2} > 0$	Minimum
Case 3		$\frac{d^2y}{dx^2} = 0$ or does not exist	Test Fails

In the event that the second derivative is zero or does not exist (Case 3), we cannot conclude whether or not an extreme exists. In this case, it may be a good idea to use the First Derivative Test at the point in question.

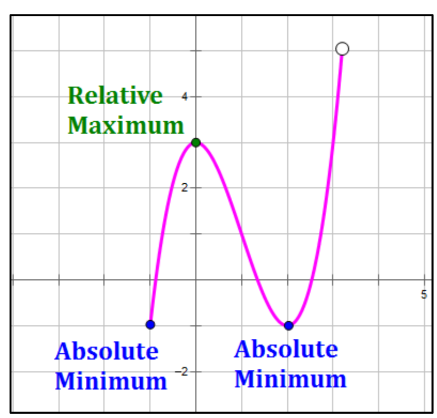
Absolute Extrema

Absolute extrema (also called "global extrema" or simply "extrema") are the highest or lowest values of the function on the interval in question. If a function is continuous, its absolute extrema exist at the locations of either its relative extrema or the endpoints of the interval.

Note that if an interval is open, the endpoint does not exist and so it cannot be an absolute extreme. This means that in some cases, a function will not have an absolute maximum or minimum on the interval in question. Discontinuities in a function can also cause a function to not have a relative maximum or minimum.

A function may have $0, 1$ or multiple absolute maxima and/or absolute minima on an interval. In the illustration to the right, the function has:

- Two absolute minima, at $(-1, -1)$ and $(2, -1)$.
- No absolute maximum (due to the discontinuity).
- One relative maximum, at $(0, 3)$.
- One relative minimum – The point located at $(2, -1)$ is both a relative minimum and an absolute minimum.



Inflection Points

Definition

An *inflection point* is a location on a curve where concavity changes from upward to downward or from downward to upward.

At an inflection point, $f''(x) = 0$ or $f''(x)$ does not exist.

However, it is not necessarily true that if $f''(x) = 0$, then there is an inflection point at $x = c$.

Testing for an Inflection Point

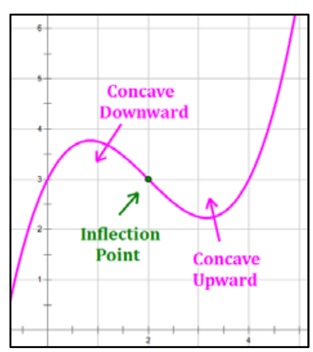
To find the inflection points of a curve in a specified interval,

- Determine all x-values ($x = c$) for which $f''(c) = 0$ or $f''(c)$ does not exist.
- Consider only c-values where the function has a tangent line.
- Test the sign of $f''(x)$ to the left and to the right of $x = c$.
- If the sign of $f''(x)$ changes from positive to negative or from negative to positive at $x = c$, then $(c, f(c))$ is an inflection point of the function.

	Second Derivative	**Sign of $\frac{d^2y}{dx^2}$ left of $x = c$**	**Sign of $\frac{d^2y}{dx^2}$ right of $x = c$**	**Inflection Point?**
Case 1	$\frac{d^2y}{dx^2} = 0$ or $\frac{d^2y}{dx^2}$ does not exist	$-$	$-$	No
Case 2		$-$	$+$	Yes
Case 3		$+$	$+$	No
Case 4		$+$	$-$	Yes

Note: inflection points cannot exist at the endpoints of a closed interval.

Concavity



A function, f, is *concave upward* on an interval if $f'(x)$ is increasing on the interval, i.e., if $f''(x) > 0$.

A function, f, is *concave downward* on an interval if $f'(c)$ is decreasing on the interval, i.e., if $f''(x) < 0$.

Concavity changes at inflection points, from upward to downward or from downward to upward. In the illustration at right, an inflection point exists at the point $(2, 3)$.

Special Case: Extrema and Inflection Points of Polynomials

For a polynomial, $f(x)$, critical values of exist at all x-values for which $f'(x) = 0$. However, critical values do not necessarily produce extrema. Possible inflection points exist at all x-values for which $f''(x) = 0$. However, not all of these x-values produce inflection points.

To find the extrema and inflection points of a polynomial we can look at the factored forms of $f'(x)$ and $f''(x)$, respectively. Every polynomial can be factored into linear terms with real roots and quadratic terms with complex roots as follows:

$$P(x) = k(x - r_1)^{a_1} \cdot (x - r_2)^{a_2} \dots (x - r_n)^{a_n} \cdot Q_1(x) \cdot Q_2(x) \dots Q_m(x)$$

where, k is a scalar (constant), each r_i is a real root of $f(x)$, each exponent a_i is an integer, and each Q_j is a quadratic term with complex roots.

Extrema

The exponents (a_i) of the linear factors of $f'(x)$ determine the existence of extrema.

- An odd exponent on a linear term of $f'(x)$ indicates that $f'(x)$ crosses the x-axis at the root of the term, so $f(x)$ has an extreme at that root. Further analysis is required to determine whether the extreme is a maximum or a minimum.
- An even exponent on a linear term of $f'(x)$ indicates that $f'(x)$ bounces off the x-axis at the root of the term, so $f(x)$ does not have an extreme at that root.

Example 3.1: Consider $f'(x) = (x+3)^3(x+2)^2\left(x+\sqrt{3}\right)^3\left(x-\sqrt{3}\right)^3(x-4)^2(x-7)$.

The original polynomial, $f(x)$, has critical values for each term: $CV = \{-3, -2, -\sqrt{3}, \sqrt{3}, 4, 7\}$.

However, extrema exist only for the terms with odd exponents: $Extrema = \{-3, -\sqrt{3}, \sqrt{3}, 7\}$.

Inflection Points (PI)

The exponents (a_i) of the linear factors of $f''(x)$ determine the existence of inflection points.

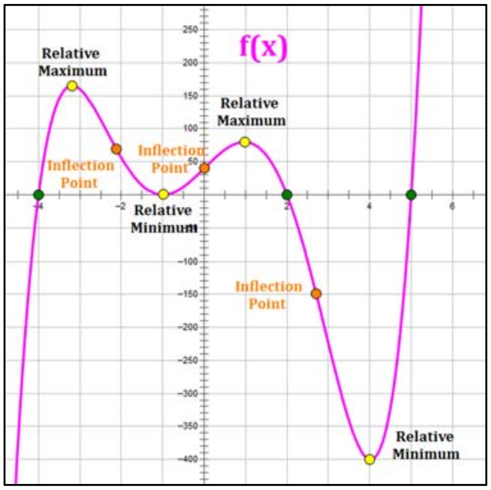
- An odd exponent on a linear term of $f''(x)$ indicates that $f(x)$ has an inflection point at the root of that term.
- An even exponent on a linear term of $f''(x)$ indicates that $f(x)$ does not have an inflection point at the root of that term.

Example 3.2: Consider $f''(x) = (x+3)^3(x+2)^2\left(x+\sqrt{3}\right)^3\left(x-\sqrt{3}\right)^3(x-4)^2(x-7)$.

Inflection points exist only for the terms with odd exponents: $PI = \{-3, -\sqrt{3}, \sqrt{3}, 7\}$.

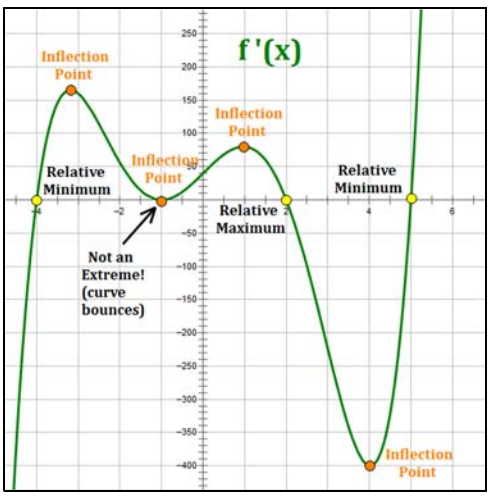
Key Points on $\mathbf{f(x), f'(x)}$ and $\mathbf{f''(x)}$ – Alauria Diagram

An **Alauria Diagram** shows a single curve as $f(x)$, $f'(x)$ or $f''(x)$ on a single page. The purpose of the diagram is to answer the question: If the given curve is $f(x)$, $f'(x)$ or $f''(x)$, where are the key points on the graph.



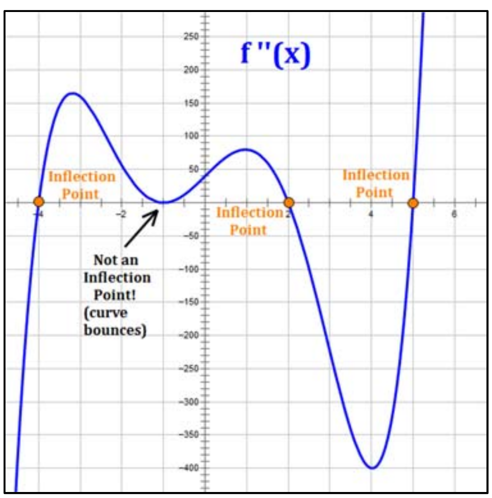
If the curve represents $f(x)$:

- $f(x)$'s x-intercepts (green and one yellow) exist where the curve touches the x-axis.
- Relative maxima and minima (yellow) exist at the tops and bottoms of humps.
- Inflection points (orange) exist where concavity changes from up to down or from down to up.



If the curve represents $f'(x)$ (1st derivative):

- $f(x)$'s x-intercepts cannot be seen.
- Relative maxima and minima of $f(x)$ (yellow) exist where the curve crosses the x-axis. If the curve bounces off the x-axis, there is no extreme at that location.
- Inflection points of $f(x)$ (orange) exist at the tops and bottoms of humps.



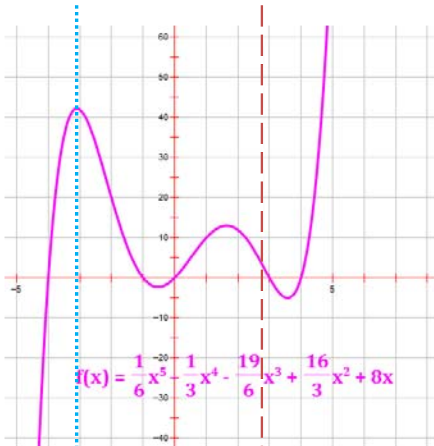
If the curve represents $f''(x)$ (2nd derivative):

- $f(x)$'s x-intercepts cannot be seen.
- Relative maxima and minima of $f(x)$ cannot be seen.
- Inflection points of $f(x)$ (orange) exist where the curve crosses the x-axis. If the curve bounces off the x-axis, there is no inflection point at that location.

Key Points on $\mathbf{f(x), f'(x)}$ and $\mathbf{f''(x)}$

The graphs below show $f(x)$, $f'(x)$ or $f''(x)$ for the same 5th degree polynomial function. The dotted blue vertical line identifies one location of an extreme (there are four, but only one is illustrated) The dashed dark red vertical line identifies one location of a point of inflection (there are three, but only one is illustrated).

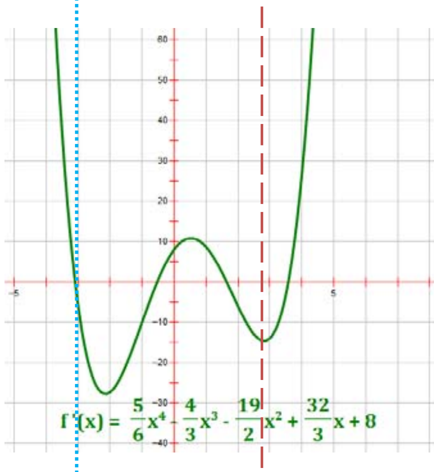
$f(x)$



In a graph of $f(x)$:

- Relative extrema exist at the tops and bottom of humps.
- Inflection points exist at locations where concavity changes from up to down or from down to up.

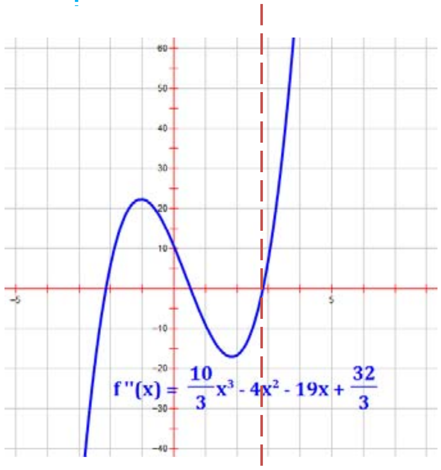
$f'(x)$



In a graph of $f'(x)$:

- Relative extrema of $f(x)$ exist where the curve crosses the x-axis. If the curve bounces off the x-axis, there is no extreme at that location.
- Inflection points of $f(x)$ exist at the tops and bottoms of humps.

$f''(x)$



In a graph of $f''(x)$:

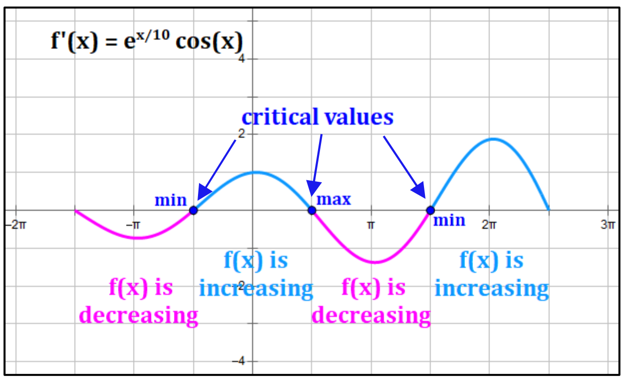
- Relative extrema of $f(x)$ cannot be seen.
- Inflection points of $f(x)$ exist where the curve crosses the x-axis. If the curve bounces off the x-axis, there is no inflection point at that location.

What Does the Graph of $f'(x)$ Tell Us about $f(x)$?

Short answer: a lot! Consider the graph of the derivative of $f(x)$ when $\boldsymbol{f'(x) = e^{x/10} \cdot \cos x}$ on the interval $\left[-\frac{3}{2}\pi, \frac{5}{2}\pi\right]$.

Increasing vs. Decreasing

We can tell if $f(x)$ is **increasing** or **decreasing** based on whether $f'(x)$ is **positive** or **negative**. Critical values exist where $f'(x)$ is zero or does not exist. Relative maxima and minima exist at critical values if the graph of $f'(x)$ crosses the x-axis. See the graph and chart below. *Note that critical values, relative maxima and relative minima do not exist at endpoints of an interval.*

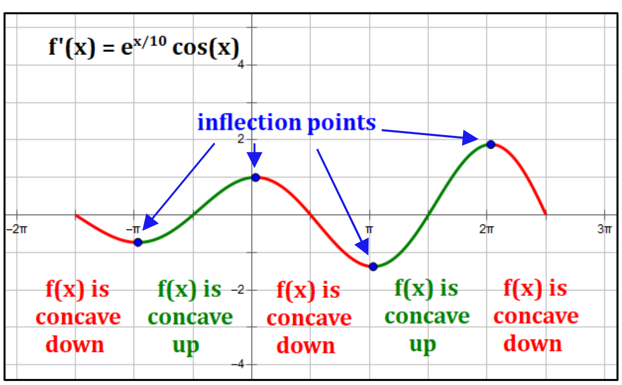


Increasing vs. Decreasing		
$f(x)$	Increasing	Decreasing
$f'(x)$	Positive	Negative

All items in a column occur simultaneously.

Concavity

We can tell if $f(x)$ is **concave up** or **concave down** based on whether $f'(x)$ is **increasing** or **decreasing**. Inflection Points exist at the extrema of $f'(x)$, i.e. at the top and bottom of any humps on the graph of $f'(x)$. See the graph and chart below. *Note that inflection points do not exist at endpoints of an interval.*



Concavity		
$f(x)$	Concave up	Concave down
$f'(x)$	Increasing	Decreasing
$f''(x)$	Positive	Negative

All items in a column occur simultaneously.

Simultaneous Behavior of $f(x)$, $f'(x)$, and $f''(x)$

A question often faced by Calculus students is: Given information about one of $f(x)$, $f'(x)$, or $f''(x)$ on a specific interval, what can be said about the behavior of the others on the same interval? For this purpose, we can use the Natalie Chart shown below:

Natalie Chart				
$f(x)$	Increasing	Decreasing	Concave Up	Concave Down
$f'(x)$	$+$	$-$	Increasing	Decreasing
$f''(x)$			$+$	$-$

Using the Natalie Chart

The Natalie Chart extends the information from the previous page into an expandible format. Information relating to the simultaneous behavior of $f(x)$, $f'(x)$, and $f''(x)$ is provided in a single column in the chart. For example:

- If we are told that $f'(x)$ is increasing on a given interval, the first magenta column in the chart tells us that $f(x)$ is concave up and $f''(x)$ is positive on the same interval.
- If we are told that $f'(x) < 0$ on a given interval, the second blue column in the chart tells us that $f(x)$ is decreasing. We cannot determine any information about the behavior of $f''(x)$ in this case, so those cells in the table are blank.

Expanding the Natalie Chart

Note that the information in the Natalie Chart is expandible to any set of three consecutive derivatives of a function by adding rows and columns.

Expanded Natalie Chart						
$f(x)$	Increasing	Decreasing	Concave Up	Concave Down		
$f'(x)$	$+$	$-$	Increasing	Decreasing	Concave Up	Concave Down
$f''(x)$			$+$	$-$	Increasing	Decreasing
$f'''(x)$					$+$	$-$

In this expanded chart, notice that knowing whether $f''(x)$ is increasing or decreasing on an interval (green text) provides information about the simultaneous behavior of $f'(x)$ and $f'''(x)$ on the same interval. Adding additional rows and columns can provide information about the simultaneous behavior of any three consecutive derivatives of any given function.

Curve Sketching

Curve Sketching is much easier with the tools of Calculus. In particular, the calculation of derivatives allows the student to identify critical values (relative maxima and minima) and inflection points for a curve. A curve can then be broken into intervals for which the various characteristics (e.g., increasing or decreasing, concave up or down) can be determined.

The acronym **DIACIDE** may help the student recall the things that should be considered in sketching curves.

DIACIDE:

- **Derivatives:** generally, the student should develop the first and second derivatives of the curve, and evaluate those derivatives at each key value (e.g., critical points, inflection points) of x.

- **Intercepts:** to the extent possible, the student should develop both x- and y-intercepts for the curve. x-intercepts occur where $f(x) = 0$. y- intercepts occur at $x = 0$.

- **Asymptotes:** vertical asymptotes should be identified so that the curve can be split into continuous sub-segments. Vertical asymptotes occur at values of x where the curve approaches $-\infty$ or $+\infty$; $f'(x)$ does not exist at these values of x. Horizontal asymptotes are covered below under the category "End Behavior."

- **Critical Values:** relative maxima and minima are locations where the curve changes from increasing to decreasing or from decreasing to increasing. They occur at "critical" x-values, where $f'(x) = 0$ or where $f'(x)$ does not exist.

- **Concavity**: concavity is determined by the value of the second derivative:

  $f''(x) < 0$ implies *downward concavity*

  $f''(x) > 0$ implies *upward concavity*

- **Inflection Points:** an inflection point is a location on the curve where concavity changes from upward to downward or from downward to upward. At an inflection point, $f''(x) = 0$ or where $f''(x)$ does not exist.

- **Domain:** the domain of a function is the set of all x-values for which a y-value exists. If the domain of a function is other than "all real numbers," care should be taken to graph only those values of the function included in the domain.

- **End Behavior:** end behavior is the behavior of a curve on the left and the right, i.e., as x tends toward $-\infty$ and $+\infty$. The curve may increase or decrease unbounded at its ends, or it may tend toward a horizontal asymptote.

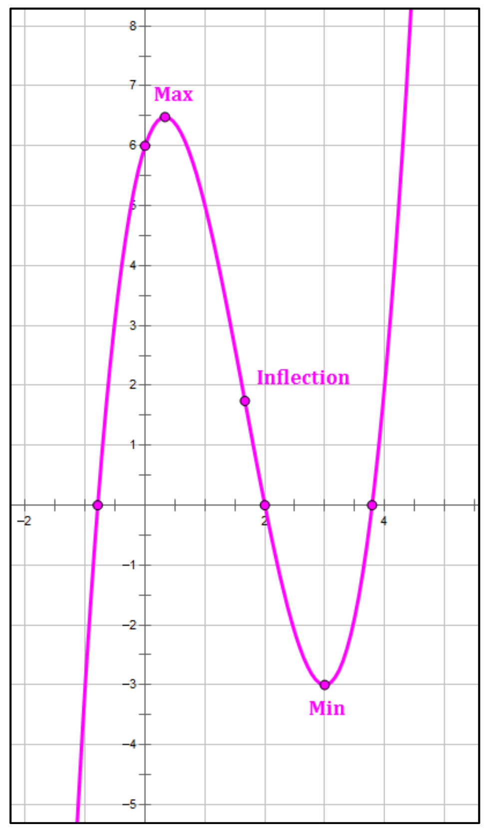
Example 3.3: Sketch the graph of $f(x) = x^3 - 5x^2 + 3x + 6$.

DIACIDE: Derivatives, **I**ntercepts, **A**symptotes, **C**ritical Values, **C**oncavity, **I**nflection Points, **D**omain, **E**nd Behavior

Note the two C's.

Derivatives: $f(x) = x^3 - 5x^2 + 3x + 6$
$f'(x) = 3x^2 - 10x + 3$
$f''(x) = 6x - 10$

Intercepts: Use synthetic division to find: $x = 2$, so: $f(x) = (x-2) \cdot (x^2 - 3x - 3)$
Then, use the quadratic formula to find: $x = \frac{3 \pm \sqrt{21}}{2} = \{-0.791, 3.791\}$
x-intercepts, then, are: $\{-0.791,\ 2,\ 3.791\}$
y-intercepts: $f(0) = 6$



Asymptotes: None for a polynomial

Critical Values: $f'(x) = 3x^2 - 10x + 3 = 0$ at $x = \left\{\frac{1}{3}, 3\right\}$
Critical Points are: $\{(.333,\ 6.481), (3,\ -3)\}$
$f''(.333) < 0$, so $(.333,\ 6.481)$ is a relative maximum
$f''(3) > 0$, so $(3,\ -3)$ is a relative minimum

Concavity: $f''(x) < 0$ for $x < 1.667$ (concave downward)
$f''(x) > 0$ for $x > 1.667$ (concave upward)

Inflection Points: $f''(x) = 6x - 10 = 0$ at $x = \frac{5}{3} \sim 1.667$
Inflection Point is: $(1.667,\ 1.741)$

Domain: All real values of x for a polynomial

End Behavior: Positive lead coefficient on a cubic equation implies that:
$\lim_{x \to -\infty} f(x) = -\infty$, and
$\lim_{x \to \infty} f(x) = \infty$

Example 3.4: Sketch the graph of $f(x) = \dfrac{10 \cdot \sin x}{e^x}$

DIACIDE:

Derivatives: $f(x) = \dfrac{10 \cdot \sin x}{e^x}$ $\quad f'(x) = \dfrac{10 \cdot (\cos x - \sin x)}{e^x}$ $\quad f''(x) = \dfrac{-20 \cdot \cos x}{e^x}$

Intercepts: x-intercept where $\sin x = 0$, so, $x = k \cdot \pi$, with k being any integer
y-intercept at $f(0) = 0$

Asymptotes: No vertical asymptotes. Horizontal asymptote at $y = 0$.

Critical Values: $f'(x) = 0$ where $\cos x = \sin x$. Critical Points exist at $x = \left\{\frac{\pi}{4} + k \cdot \pi,\ k \in Z\right\}$
$(.707,\ 3.224)$ is a relative maximum; $(3.927, -0.139)$ is a relative minimum

There are an infinite number of relative maxima and minima, alternating at x-values that are π apart.

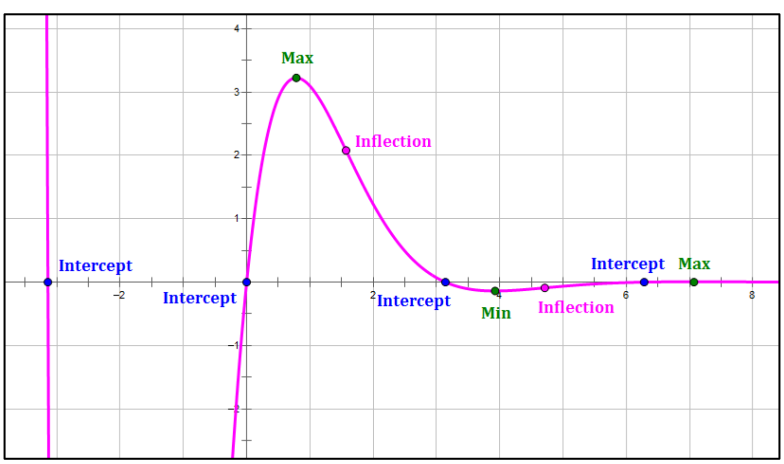
Concavity: The function is concave up where $\cos x < 0$, i.e., Quadrants II and III and is concave down where $\cos x > 0$, i.e., Quadrants I and IV.

Inflection Points: $f''(x) = 0$ where $\cos x = 0$

Inflection Points exist at: $x = \left\{\frac{\pi}{2} + k \cdot \pi,\ k \in Z\right\}$

Domain: All real values of x

End Behavior: $\lim\limits_{x \to -\infty} f(x)$ does not exist, as the function oscillates up and down with each period
$\lim\limits_{x \to \infty} f(x) = 0$



Example 3.5: Sketch the graph of $f(x) = \frac{x^2-4}{x^2-9}$

DIACIDE:

Derivatives: $f(x) = \frac{x^2-4}{x^2-9}$ $\quad f'(x) = \frac{-10x}{(x^2-9)^2}$ $\quad f''(x) = \frac{30 \cdot (x^2+3)}{(x^2-9)^3}$

Intercepts: x-intercept where $x^2 - 4 = 0$, so, $\boldsymbol{x = \pm 2}$
y-intercept at $\boldsymbol{f(0) = \frac{4}{9}}$

Plot these intercepts on the graph.

Asymptotes: Vertical asymptotes where: $x^2 - 9 = 0$, so $\boldsymbol{x = \pm 3}$.
Horizontal asymptote at:

$$y = \lim_{x\to\infty} \frac{x^2-4}{x^2-9} = \lim_{x\to-\infty} \frac{x^2-4}{x^2-9} = 1$$

Plot the asymptotes on the graph.

Critical Values: $f'(x) = 0$ where $x = 0$; so $f'(0) = 0$
Since $f''(0) = -\frac{10}{81} < 0$, $\left(\mathbf{0}, \frac{\mathbf{4}}{\mathbf{9}}\right)$ is a relative maximum

Plot the critical values on the graph.

Concavity: The concavity of the various intervals are shown in the table on the next page

Inflection Points: $f''(x) = 0$ where $x^2 + 3 = 0$
Therefore, there are no real inflection points

If there are inflection points, plot them on the graph.

Domain: All real values of x, except at the vertical asymptotes
So, the domain is: All Real $x \neq \{-3, 3\}$

End Behavior: $\lim_{x\to-\infty} \frac{x^2-4}{x^2-9} = 1$

$\lim_{x\to\infty} \frac{x^2-4}{x^2-9} = 1$

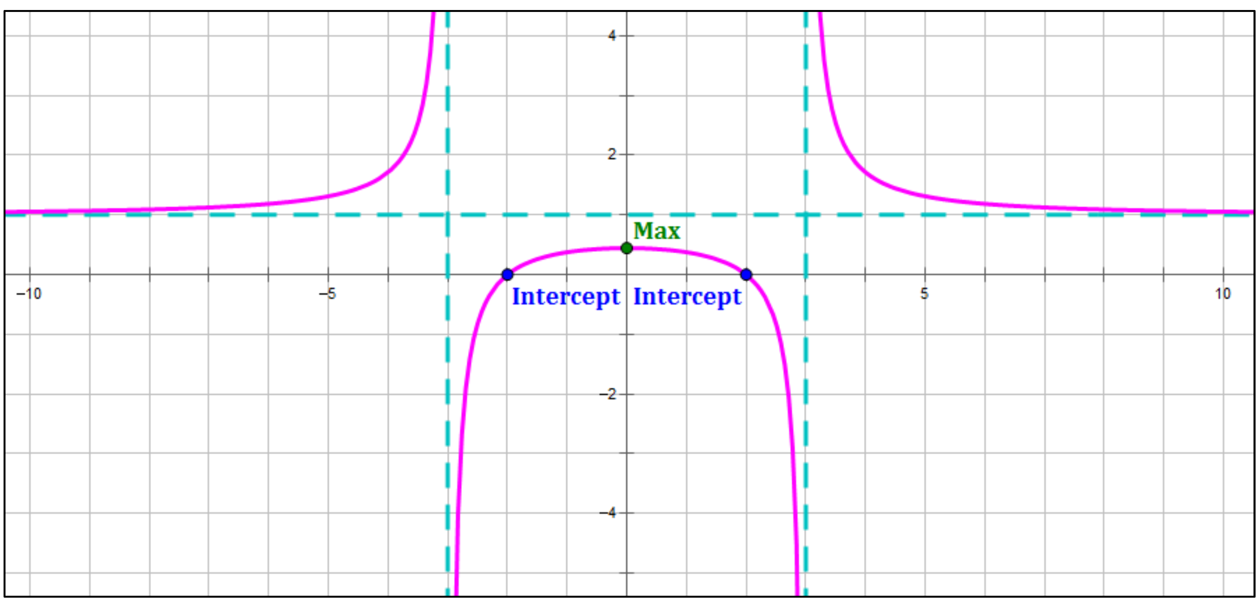
These imply the existence of a horizontal asymptote at $y = 1$.

Example 3.5 (cont'd)

In some cases, it is useful to set up a table of intervals which are defined by the key values identified in **green** above: $x = \{-3, 0, 3\}$. The key values are made up of:

- Vertical asymptotes
- Relative maxima and minima
- Inflection Points

x-values	$f(x)$	$f'(x)$	$f''(x)$	Graph Characteristics
$(-\infty, -3)$		$+$	$+$	curve increasing, concave up
-3	undefined	undefined	undefined	vertical asymptote
$(-3, 0)$		$+$	$-$	curve increasing, concave down
0	$.444$	0	$-$	relative maximum
$(0, 3)$		$-$	$-$	curve decreasing, concave down
3	undefined	undefined	undefined	vertical asymptote
$(3, \infty)$		$-$	$+$	curve decreasing, concave up



Determining the Shape of a Curve Based On Its Derivatives

The possible shapes of a curve, based on its first and second derivatives are:

Increasing function	Decreasing function	Increasing function	Decreasing function
$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$	$f'(x) < 0$
Concave up	Concave up	Concave down	Concave down
$f''(x) > 0$	$f''(x) > 0$	$f''(x) < 0$	$f''(x) < 0$

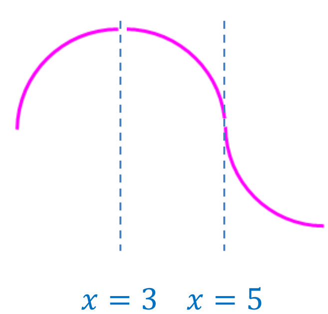
So, given a differentiable function with first and second derivatives identified, we need only match the shapes above to the intervals of the function and then join them together. If we are given points on the curve, we must also fit the shape through the given points.

Example 3.6: Suppose we want to determine the approximate shape of the curve of the differentiable function defined by the following table.

x	$1 < x < 3$	$x = 3$	$3 < x < 5$	$x = 5$	$5 < x < 7$
$f'(x)$	Positive	0	Negative	Negative	Negative
$f''(x)$	Negative	Negative	Negative	0	Positive
Curve Shape		Flat – Relative Maximum		Point of Inflection	

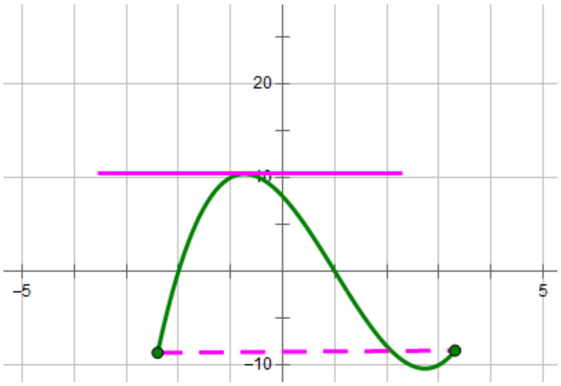
To get the shape of the function over the given interval, join the shapes for each subinterval together as shown at right.

Note: If we are given points on the curve, we must also stretch or compress the various parts of the resulting shape to fit through the given points.



Rolle's Theorem and the Mean Value Theorem

Rolle's Theorem



If

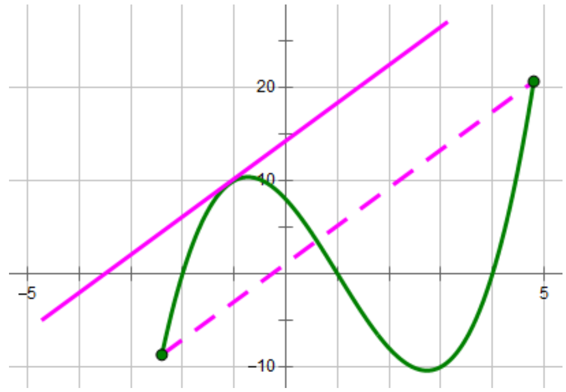
1. $f(x)$ is continuous on $[a, b]$.
2. $f(x)$ is differentiable in (a, b).
3. $f(a) = f(b)$.

Then

There is at least one value c in (a, b) such that $f'(c) = 0$.

Conclusion in Words: There is at least one point in (a, b) with a horizontal tangent line.

Mean Value Theorem (MVT)



If

1. $f(x)$ is continuous on $[a, b]$.
2. $f(x)$ is differentiable in (a, b).

Then

There is at least one value c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Conclusion in Words: There is at least one point in (a, b) where the slope of the tangent line has the same slope as the secant line over $[a, b]$.

Note: If the conditions for Rolle's Theorem are satisfied, then either Rolle's Theorem or the MVT can be applied.

- *Rolle's Theorem concludes that there is a value c such that:* $f'(c) = 0$
- *The MVT concludes that there is a value c such that:* $f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{0}{b-a} = 0$
- *These two conclusions are identical.*

Related Rates

Related Rates Problems

To solve problems that involve rates of change of two or more related variables, each with respect to a third variable, we must take derivatives with respect to the third variable (often, *time*) and remember to use the chain rule at each step. There are numerous methods that can be used to solve these problems; one that students have found particularly helpful is described and illustrated below.

The General-Specific Method

This method breaks up the solution into the General and Specific Cases described in the problem, as follows:

The General Case

- Deal with all variables in the abstract, without any numbers substituted for the variables.
- Set up any formulas required to solve the problem (e.g., volume of a cone).
- Take any derivatives (based on the above formulas) required to solve the problem.

The Specific Case

- Record any values of variables for the specific situation described in the problem.
- Calculate any additional values required based on those provided in the problem (e.g., the length of the third side of a right triangle).
- After any derivatives are developed in the General Case, substitute values for the variables in the derivative equation.
- Derive the solution to the problem by solving the resulting equation.

Notes:

- For some problems, you may need to draw a picture of the situation described in the problem. In these problems, you should draw a picture for the General Case and a second picture for the Specific Case. See Example 3.9, below.
- In the examples that follow, the General Case is shown on the left and the Specific Case is shown on the right.

Example 3.7: The volume of a cylinder is changing by 48 cm^3 per second when the radius of the cylinder is 2 cm. If the height is twice the radius, find the rate of change of the radius when r = 2 cm. Note: $V = \pi r^2 h$.

General Case

We are asked to find $\frac{dr}{dt}$:

$h = 2r$

$V = \pi r^2 h = \pi r^2 (2r) = 2\pi r^3$

Take the derivatives of both sides with respect to t:

$$\frac{dV}{dt} = 6\pi r^2 \frac{dr}{dt}$$

After this part is done, move to the Specific Case.

Specific Case

Information given:

$\frac{dV}{dt} = 48$ $\quad r = 2$

Substitute values into the equation derived in the General Case:

$$\frac{dV}{dt} = 6\pi r^2 \frac{dr}{dt}$$

$$48 = 6\pi \cdot 2^2 \cdot \frac{dr}{dt}$$

Do some algebra to calculate:

$$\boldsymbol{\frac{dr}{dt}} = \frac{48}{24\pi} = \boldsymbol{\frac{2}{\pi}}\ \mathrm{cm/sec}$$

Example 3.8: The SA of a sphere is changing by 36 cm^2 per second when the radius of the cylinder is 3 *cm*. Find the rate of change of the radius when *r* = 3 *cm*. Note: $SA = 4\pi r^2$.

General Case

We are asked to find $\frac{dr}{dt}$:

$SA = 4\pi r^2$

Take the derivatives of both sides with respect to t:

$$\frac{dSA}{dt} = 8\pi r \frac{dr}{dt}$$

After this part is done, move to the Specific Case.

Specific Case

Information given:

$\frac{dSA}{dt} = 36$ $\quad r = 3$

Substitute values into the equation derived in the General Case:

$$\frac{dSA}{dt} = 8\pi r \frac{dr}{dt}$$

$$36 = 8\pi \cdot 3 \cdot \frac{dr}{dt}$$

Do some algebra to calculate:

$$\boldsymbol{\frac{dr}{dt}} = \frac{36}{24\pi} = \boldsymbol{\frac{3}{2\pi}}\ \mathrm{cm/sec}$$

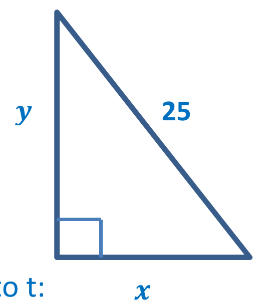
Example 3.9: A ladder 25 feet long is leaning against the wall of a house. The base of the ladder is pulled away from the wall at a rate of 2 feet per second. How fast is the top of the ladder moving down the wall when its base is 7 feet from the wall?

General Case

We are asked to find $\frac{dy}{dt}$

Based on the drawing:

$$x^2 + y^2 = 625$$



Take the derivatives of both sides with respect to t:

$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$$

After this part is done, move to the Specific Case.

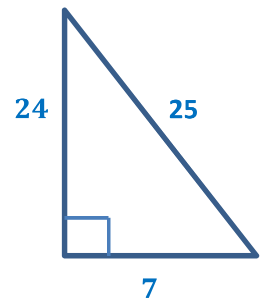
Specific Case

Information given:

$$x = 7$$

Calculate: $y = 24$

$$\frac{dx}{dt} = 2$$



Substitute values into the equation derived in the General Case:

$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$$

$$2 \cdot 7 \cdot 2 + 2 \cdot 24 \cdot \frac{dy}{dt} = 0$$

Do some algebra to calculate:

$$\frac{dy}{dt} = -\frac{28}{48} = -\frac{7}{12} = -0.583\,\frac{\text{ft}}{\text{sec}}$$

Example 3.10: The radius *r* of a circle is increasing at a rate of 3 cm/minute. Find the rate of change of the area when the circumference $C = 12\pi$ cm.

General Case

We are asked to find $\frac{dA}{dt}$:

$$A = \pi r^2$$

Take the derivatives of both sides with respect to t:

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

After this part is done, move to the Specific Case.

Specific Case

Information given:

$$C = 2\pi r = 12\pi \qquad \frac{dr}{dt} = 3$$

Substitute values into the equation derived in the General Case:

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\frac{dA}{dt} = C \cdot \frac{dr}{dt} = 12\pi \cdot 3 = 36\pi\,\frac{\text{cm}^2}{\text{min}}$$

Kinematics (Particle Motion)

Position

Position is the location of a particle at a point in time. It is typically represented by the functions $s(t)$ or $x(t)$.

Displacement

Displacement is a measure of the difference between a particle's starting point and its ending point. It may be either positive or negative. A formula for displacement is:

$\Delta s = s - s_0$, where s is the position at any point in time, and s_0 is the starting position.

Distance

Distance is a measure of the total movement of a particle; it is always a positive value. Total distance is the sum of the absolute values of the displacements of a particle in its various directions.

Example 3.11: A particle moves from $x = 0$ to $x = 6$ to $x = 2$.

- Displacement $= end - start = 2 - 0 = 2$ units
- Distance = sum of absolute values of individual displacements

  $= |6 - 0| + |2 - 6| = 10$ units

Velocity

Velocity measures the rate of change in position. Instantaneous velocity is generally shown using the variable v and average velocity is generally shown as $\bar{v}$. Velocity may also be shown as a vector $\vec{\mathbf{v}}$, which has both magnitude and direction. The following formulas apply to velocity:

Instantaneous velocity: $v = \dfrac{ds}{dt}$ (i.e, the derivative of the position function)

Velocity at time t: $v = v_0 + at$ (where, v_0 is initial velocity and a is a constant acceleration)

Average velocity: $\bar{v} = \dfrac{\text{total displacement}}{\text{total time}} = \dfrac{\Delta s}{\Delta t} = \dfrac{s(t_2) - s(t_1)}{t_2 - t_1}$

Velocity may be either positive or negative.

Speed

Speed, like velocity, measures the rate of change in position. However, unlike velocity, speed is always positive (it does not have direction). Instantaneous speed is the absolute value of velocity $|v|$ at a point in time. Average speed is based on distance instead of displacement. The following formulas apply to speed:

Instantaneous speed: $|v| = \left|\dfrac{ds}{dt}\right|$ (i.e, the absolute value of the velocity function)

Average speed: $\dfrac{\text{total distance}}{\text{total time}}$

A note about speed:

- Speed is increasing when velocity and acceleration have the same sign (either $+$ or $-$).
- Speed is decreasing when velocity and acceleration have different signs (one $+$, one $-$).

Acceleration

Acceleration measures the rate of change in velocity. Instantaneous acceleration is generally shown using the variable a and average acceleration is generally shown as $\bar{a}$. Acceleration may also be shown as a vector $\vec{\mathbf{a}}$, which has both magnitude and direction. The following formulas apply to acceleration:

Instantaneous acceleration: $a = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2}$

Average acceleration: $\bar{a} = \dfrac{\text{total change in velocity}}{\text{total time}} = \dfrac{\Delta v}{\Delta t} = \dfrac{v(t_2)-v(t_1)}{t_2-t_1}$

Moving Among Functions

The following diagram describes how to move back and forth among the position, velocity and acceleration functions. (Note: integration is handled in a subsequent chapter.)

$$\mathit{Position}\quad \underset{\xleftarrow[\mathit{Integrate}]{}}{\xrightarrow{\mathit{Differentiate}}}\quad \mathit{Velocity}\quad \underset{\xleftarrow[\mathit{Integrate}]{}}{\xrightarrow{\mathit{Differentiate}}}\quad \mathit{Acceleration}$$

Summary of Kinematics Terms

Consider the following function definitions in relation to a particle in motion:

- $s(t)$ or $x(t)$ is the position function along the x-axis.
- $v(t) = s'(t)$ is the velocity function along the x-axis.
- $speed(t) = |v(t)|$ is the speed function along the x-axis.
- $a(t) = v'(t) = s''(t)$ is the acceleration function along the x-axis.

Then the following terms relate to the functions defined above:

	Term/Description	Meaning
	Initially	$t = 0$
	At the origin	$s(t) = 0$
velocity	At rest (i.e., zero velocity)	$v(t) = 0$
	Positive velocity (moving to the right)	$v(t) > 0$
	Negative velocity (moving to the left)	$v(t) < 0$
	Average velocity (or the approximation of velocity over an interval $[a, b]$)	$\bar{v} = \frac{\Delta\text{ position}}{\Delta\text{ time}} = \frac{s(b) - s(a)}{b - a}$
	Instantaneous velocity at time $t = c$	$v(c) = s'(c)$
	Particle changes directions	$v(t) = 0$, changes signs at time t
speed	Speed is increasing (particle is accelerating)	$v(t), a(t)$ have the same sign (+ or -)
	Speed is decreasing (particle is decelerating)	$v(t), a(t)$ have different signs
acceleration	Positive acceleration	$a(t) > 0$
	Negative acceleration	$a(t) < 0$
	Average acceleration (or the approximation of acceleration over an interval $[a, b]$)	$\bar{a} = \frac{\Delta\text{ velocity}}{\Delta\text{ time}} = \frac{v(b) - v(a)}{b - a}$
	Instantaneous acceleration at time $t = c$	$a(c) = v'(c) = s''(c)$

Differentials

Finding the Tangent Line

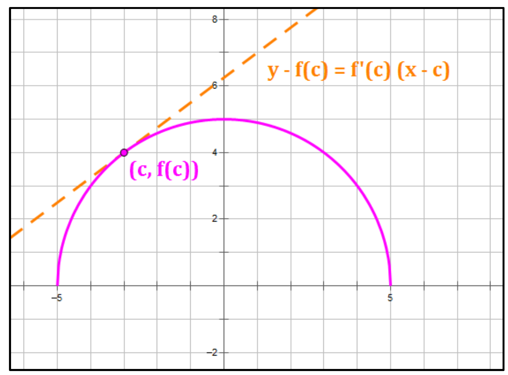
Most problems that use differential to find the tangent line deal with three issues:

- Developing the equation of a tangent line at a point on a curve
- Estimating the value of a function using the tangent line.
- Estimating the change in the values of a function between two points, using the tangent line.

In each case, the tangent line is involved, so let's take a look at it. The key equation is:

$$y = f(c) + f'(c) \cdot (x - c)$$

How does this equation come about? Let's look at a curve and find the equation of the tangent line to that curve, in the general case. See the diagram below:



- Let our point on the curve be $\left(c, f(c)\right)$.
- The slope of the tangent line at $\left(c, f(c)\right)$ is $f'(c)$.
- Use the point-slope form of a line to calculate the equation of the line:

  $y - y_1 = m(x - x_1) \quad \Rightarrow \quad y - f(c) = f'(c) \cdot (x - c)$
- Add $f(c)$ to both sides of the equation to obtain the form shown above

Let's take a closer look at the pieces of the equation:

First, define your anchor, c, and calculate $f(c)$ and $f'(c)$. Substitute these into the equation and you are well on your way to a solution to the problem.

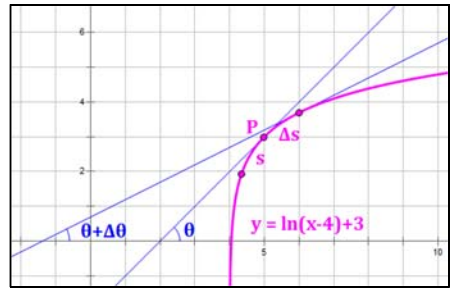
$(x - c)$ is also shown as Δx. It is the difference between the x-value you are evaluating and your anchor to the curve, which is the tangent point $\left(c, f(c)\right)$.

$$y = f(c) + \underbrace{f'(c) \cdot \overbrace{(x - c)}}$$

This is the "change part". So, when you are asked about the change in $f(x)$ between two points or the potential error in measuring something, this is the part to focus on.

Curvature

Curvature is the rate of change of the direction of a curve at a point, P (i.e., how fast the curve is turning at point P). Direction is based on $\boldsymbol{\theta}$**,** the angle between the x-axis and the tangent to the curve at P. The rate of change is taken with respect to $\boldsymbol{s}$, the length of an arbitrary arc on the curve near point P. We use the Greek letter kappa, $\boldsymbol{\kappa}$, for the measure of curvature.



This is illustrated for the function $y = \ln(x-4)+3$ at right.

$$\kappa = \frac{d\theta}{ds} = \lim_{\Delta s \to 0} \frac{\Delta\theta}{\Delta s}$$

This results in the following equations for $\boldsymbol{\kappa}$**:**

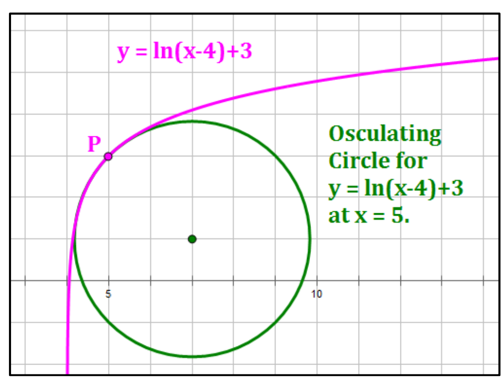
$$\kappa = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}} \qquad \text{or} \qquad \kappa = \frac{-\frac{d^2x}{dy^2}}{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}$$

Polar Form: Let $r(\theta)$ be a function in polar form. Then, the polar form of curvature is given by:

$$\kappa = \frac{r^2+2(r')^2-r\,r''}{[r^2+(r')^2]^{3/2}} \qquad \text{where,} \quad r' = \frac{dr}{d\theta}, \quad r'' = \frac{d^2r}{d\theta^2}$$

The **Osculating Circle** of a curve at Point P is the circle which is:

- Tangent to the curve at point P.
- Lies on the concave side of the curve at point P.
- Has the same curvature as the curve at point P.

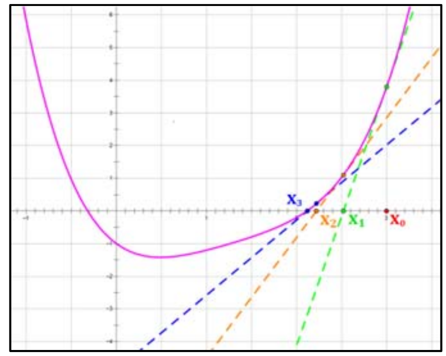


The **Radius of Curvature** of a curve at Point P is the radius of the osculating circle at point P. $R = \dfrac{1}{|\kappa|}$

The **Center of Curvature** of a curve at Point P is the center of the osculating circle at Point P.

Newton's Method

Sometimes it is difficult or impossible to find the exact roots of an equation. In such cases, approximate values may be found using numerical methods. Newton's Method is a popular approach for determining roots this way, primarily because it is simple and easily programmed for use with a computer.



Newton's Method

Use the following steps to identify a root of a function $f(x)$ using Newton's Method.

1. **Select an estimate of the root** you are looking for. Call this estimate $\boldsymbol{x_0}$. It may be useful to graph the function for this purpose.
2. **Use the differential formula (see above) to refine your estimate of the root:**

   $$y = f(x_0) + f'(x_0) \cdot (x - x_0)$$

   We want an estimate of x when $y = 0$. Setting $y = 0$, the differential formula can be manipulated algebraically to get:

   $$x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

   Let this value of x be our next estimate, $\boldsymbol{x_1}$, of the value of the root we seek. Then,

   $$\boldsymbol{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

3. **Repeat the process to get subsequent values of $\boldsymbol{x_n}$**, i.e.,

   $$\boldsymbol{x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}}$$

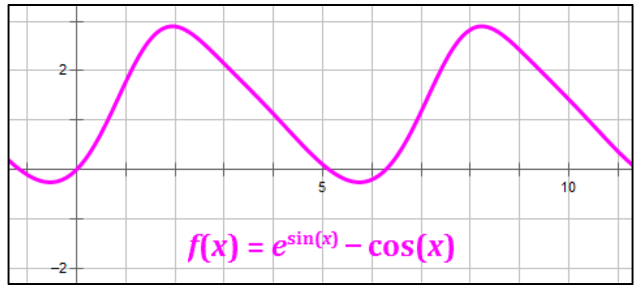
4. **Continue Step 3 until the sequence $\boldsymbol{\{x_n\}}$ converges**; that is, until successive estimates round to the same value based on a predetermined level of accuracy.

When Newton's Method Diverges

Newton's Method diverges under certain conditions. That is, for some functions and/or starting $\boldsymbol{x_0}$ values, successive values of $\boldsymbol{x_n}$ may not exist, may fluctuate back and forth between values, or may grow further and further away from the initial estimate of the root. When this occurs, you may want to select a different starting value of $\boldsymbol{x_0}$ and try again. However, the student should be aware that there are situations where Newton's Method fails altogether.

Example 3.12: Estimate the root of $f(x) = e^{\sin x} - \cos x$ near $x = 5$ to six decimal places.

Let's graph the function. In the graph, it is clear that there is a root close to $x = 5.$ So, we are hopeful that Newton's Method will converge quickly.



We begin with the following:

- $x_0 = 5$
- $f(x) = e^{\sin x} - \cos x$
- $f'(x) = \left(e^{\sin x} \cdot \cos x\right) + \sin x$

Now, let's develop successive values of x_n. *Note: Microsoft Excel is useful for this purpose.*

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 5 - \frac{e^{\sin 5} - \cos 5}{(e^{\sin 5} \cdot \cos 5) + \sin 5} = 5 - \frac{0.099643}{-0.8502} = 5.1172$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 5.1172 - \frac{e^{\sin 5.1172} - \cos 5.1172}{(e^{\sin 5.1172} \cdot \cos 5.1172) + \sin 5.1172} = 5.123764$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 5.123764 - \frac{e^{\sin 5.123764} - \cos 5.123764}{(e^{\sin 5.123764} \cdot \cos 5.123764) + \sin 5.123764} = 5.123787$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 5.123787 - \frac{e^{\sin 5.123787} - \cos 5.123787}{(e^{\sin 5.123787} \cdot \cos 5.123787) + \sin 5.123787} = \mathbf{5.123787}$$

At this point, we stop the process because $x_4 = x_3$ when rounded to six decimals. The sequence of $\{x_n\}$ appears to have converged to **5.123787,** which is our solution. If you like, you can use a modern graphing calculator to verify that this is in fact a good estimate of the desired root of $f(x)$.

Note: While the use of modern handheld graphing calculators makes Newton's Method unnecessary in the Calculus classroom, its use in mathematical computer applications is essential. It is very useful in Microsoft Excel, Visual Basic, Python, Java and other applications in which the determination of a root is automated.

Rules of Indefinite Integration

Note: the rules presented in this chapter omit the " $+$ C " term that must be added to all indefinite integrals in order to save space and avoid clutter. Please remember to add the " $+$ C " term on all work you perform with indefinite integrals.

Basic Rules

$$\int c\,du = cu$$

$$\int c\,f(u)\,du = c\int f(u)\,du$$

$$\int f(u) + g(u)\,du = \int f(u)\,du + \int g(u)\,du$$

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

Power Rule

$$\int (u^n)\,du = \frac{1}{n+1}\cdot u^{n+1} \quad (n \neq 1) \qquad \int \frac{1}{u}\,du = \ln|u|$$

Exponential and Logarithmic Functions $(a > 0, a \neq 1)$

$$\int e^u\,du = e^u$$

$$\int a^u\,du = \frac{1}{\ln a}\,a^u \qquad \int \frac{1}{u\ln u}du = \ln(\ln u)$$

Integrals of Special Functions

Exponential and Logarithmic Functions

$\int e^x\,dx = e^x$	$\int e^u\,du = e^u$
$\int a^x\,dx = \dfrac{a^x}{\ln a}$	$\int a^u\,du = \dfrac{a^u}{\ln a}$
$\int \dfrac{1}{x}\,dx = \ln\lvert x\rvert$	$\int \dfrac{1}{u}\,du = \ln\lvert u\rvert$
$\int \ln x\,dx = x\ln x - x$	$\int \ln u\,dx = u\ln u - u$
$\int \dfrac{1}{x\ln x}\,dx = \ln(\ln x)$	$\int \dfrac{1}{u\ln u}\,du = \ln(\ln u)$

Trigonometric Functions

$\int \sin u\,du = -\cos u$	
$\int \cos u\,du = \sin u$	
$\int \tan u\,du = \ln\lvert\sec u\rvert = -\ln\lvert\cos u\rvert$	$\int \sec^2 u\,du = \tan u$
$\int \cot u\,du = -\ln\lvert\csc u\rvert = \ln\lvert\sin u\rvert$	$\int \csc^2 u\,du = -\cot u$
$\int \sec u\,du = \ln\lvert\sec u + \tan u\rvert$	$\int \sec u\tan u\,du = \sec u$
$\int \csc u\,du = -\ln\lvert\csc u + \cot u\rvert$	$\int \csc u\cot u\,du = -\csc u$

Derivations of the Integrals of Trigonometric Functions

$\int \tan x \, dx$

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$$

Let: $u = \cos x$ so that: $du = -\sin x \, dx$ Then,

$$\int \tan x \, dx = -\int \frac{1}{u} \, du = -\ln|u| + C = -\ln|\cos x| + C$$

$\int \cot x \, dx$

$$\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx$$

Let: $u = \sin x$ so that: $du = \cos x \, dx$ Then,

$$\int \cot x \, dx = \int \frac{1}{u} \, du = \ln|u| + C = -\ln|\sin x| + C$$

$\int \sec x \, dx$

Multiply the numerator and denominator by: $(\sec x + \tan x)$

Then,

$$\int \sec x \, dx = \int \sec x \cdot \frac{(\sec x + \tan x)}{(\sec x + \tan x)} dx = \int \frac{(\sec^2 x + \sec x \tan x)}{(\sec x + \tan x)} dx$$

Let: $u = (\sec x + \tan x)$ so that: $du = (\sec x \tan x + \sec^2 x) \, dx$

Then,

$$\int \sec x \, dx = \int \frac{1}{u} \, du = \ln|u| + C = \ln|\sec x + \tan x| + C$$

Derivations of the Integrals of Trig Functions (cont'd)

$\int \csc x \, dx$

Multiply the numerator and denominator by: $(\csc x + \cot x)$

Then,

$$\int \csc x \, dx = \int \csc x \cdot \frac{(\csc x + \cot x)}{(\csc x + \cot x)} dx = \int \frac{(\csc^2 x + \csc x \cot x)}{(\csc x + \cot x)} dx$$

Let: $u = (\csc x + \cot x)$ so that: $du = (-\csc x \tan x - \csc^2 x) \, dx$

Then,

$$\int \csc x \, dx = -\int \frac{1}{u} \, du = -\ln|u| + C = -\ln|\csc x + \cot x| + C$$

Integration Involving Inverse Trig Functions

Key Formulas:

Base Formulas

$$\int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1} u + C$$

$$\int \frac{1}{1+u^2}\,du = \tan^{-1} u + C$$

$$\int \frac{1}{u\sqrt{u^2-1}}\,du = \sec^{-1}|u| + C$$

General Formulas

$$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{1}{u\sqrt{u^2-a^2}}\,du = \frac{1}{a}\sec^{-1}\left(\frac{|u|}{a}\right) + C$$

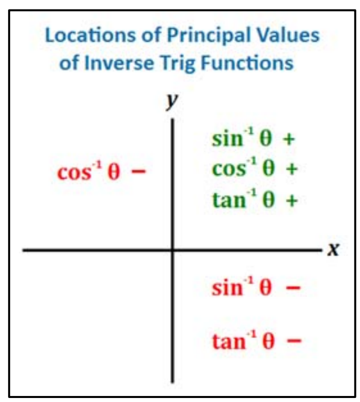
About Inverse Trig Functions

As an example, $\sin^{-1} x$ asks the question, what angle (in radians) has a sine value of x? So,

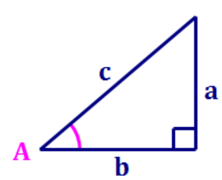
$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} \qquad \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4} \qquad \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

It is important, for these purposes, to understand the quadrants in which each inverse Trig function is defined, as shown in the following charts.

Ranges of Inverse Trigonometric Functions	
Function	**Gives a Result In:**
$\sin^{-1}\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
$\cos^{-1}\theta$	$0 \le \theta \le \pi$
$\tan^{-1}\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$



Solutions to problems involving inverse Trig functions may be expressed multiple ways. For example, in the triangle at right with sides a, b and c, the measure of angle A can be expressed as follows:



$$m\angle A = \sin^{-1}\left(\frac{a}{c}\right) = \tan^{-1}\left(\frac{a}{b}\right) = \sec^{-1}\left(\frac{c}{b}\right)$$

Some calculators will never give results using the $\sec^{-1}$ function, preferring to use the $\tan^{-1}$ function instead; the answers are equivalent. For example, $\sec^{-1}|2x| = \tan^{-1}\sqrt{4x^2-1}$.

Indefinite Integrals of Inverse Trigonometric Functions

Inverse Trigonometric Functions

$$\int \sin^{-1} u\, du = u\ \sin^{-1} u + \sqrt{1-u^2}$$

$$\int \cos^{-1} u\, du = u\ \cos^{-1} u - \sqrt{1-u^2}$$

$$\int \tan^{-1} u\, du = u\ \tan^{-1} u - \frac{1}{2}\ \ln(u^2+1)$$

$$\int \cot^{-1} u\, du = u\ \cot^{-1} u + \frac{1}{2}\ \ln(u^2+1)$$

$$\int \sec^{-1} u\, du = u\ \sec^{-1} u - \ln\left(u+\sqrt{u^2-1}\right) \qquad \sec^{-1} u \in \left\{\left(0, \frac{\pi}{2}\right)\right\}$$

$$= u\ \sec^{-1} u + \ln\left(u+\sqrt{u^2-1}\right) \qquad \sec^{-1} u \in \left\{\left(\frac{\pi}{2}, \pi\right)\right\}$$

$$\int \csc^{-1} u\, du = u\ \csc^{-1} u + \ln\left(u+\sqrt{u^2-1}\right) \qquad \csc^{-1} u \in \left\{\left(0, \frac{\pi}{2}\right)\right\}$$

$$= u\ \csc^{-1} u - \ln\left(u+\sqrt{u^2-1}\right) \qquad \csc^{-1} u \in \left\{\left(-\frac{\pi}{2}, 0\right)\right\}$$

Involving Inverse Trigonometric Functions

$$\int \frac{1}{\sqrt{1-u^2}}\ du = \sin^{-1} u \qquad \int \frac{1}{\sqrt{a^2-u^2}}\ du = \sin^{-1}\left(\frac{u}{a}\right)$$

$$\int \frac{1}{1+u^2}\ du = \tan^{-1} u \qquad \int \frac{1}{a^2+u^2}\ du = \frac{1}{a}\ \tan^{-1}\left(\frac{u}{a}\right)$$

$$\int \frac{1}{u\sqrt{u^2-1}}\ dx = \sec^{-1}|u| \qquad \int \frac{1}{u\sqrt{u^2-a^2}}\ dx = \frac{1}{a}\ \sec^{-1}\left(\frac{|u|}{a}\right)$$

Integrals of Special Functions
Selecting the Right Function for an Integral

<table>
<tr><th>Form</th><th>Function</th><th>Integral</th></tr>
<tr><td>$\int \frac{1}{\sqrt{a^2-u^2}}\,du$</td><td>$\sin^{-1} u$</td><td>$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right)$</td></tr>
<tr><td>$\int \frac{1}{a^2+u^2}\,du$</td><td>$\tan^{-1} u$</td><td>$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)$</td></tr>
<tr><td>$\int \frac{1}{u\sqrt{u^2-a^2}}\,dx$</td><td>$\sec^{-1} u$</td><td>$\int \frac{1}{u\sqrt{u^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{|u|}{a}\right)$</td></tr>
<tr><td>$\int \frac{1}{\sqrt{u^2+a^2}}\,du$</td><td>$\sinh^{-1} u$ *</td><td>$\int \frac{1}{\sqrt{u^2+a^2}}\,du = \ln\left(u+\sqrt{u^2+a^2}\right)$</td></tr>
<tr><td>$\int \frac{1}{\sqrt{u^2-a^2}}\,du$</td><td>$\cosh^{-1} u$ *</td><td>$\int \frac{1}{\sqrt{u^2-a^2}}\,du = \ln\left(u+\sqrt{u^2-a^2}\right)$</td></tr>
<tr><td>$\int \frac{1}{a^2-u^2}\,du\,\Big|_{a>u}$</td><td>$\tanh^{-1} u$ *</td><td rowspan="2">$\int \frac{1}{a^2-u^2}\,du = \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right|$</td></tr>
<tr><td>$\int \frac{1}{u^2-a^2}\,du\,\Big|_{u>a}$</td><td>$\coth^{-1} u$ *</td></tr>
<tr><td>$\int \frac{1}{u\sqrt{a^2-u^2}}\,du$</td><td>$\operatorname{sech}^{-1} u$ *</td><td>$\int \frac{1}{u\sqrt{a^2-u^2}}\,du = -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2-u^2}}{|u|}\right)$</td></tr>
<tr><td>$\int \frac{1}{u\sqrt{a^2+u^2}}\,du$</td><td>$\operatorname{csch}^{-1} u$ *</td><td>$\int \frac{1}{u\sqrt{a^2+u^2}}\,du = -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2+u^2}}{|u|}\right)$</td></tr>
</table>

* This is an inverse hyperbolic function. For more information, see Chapter 6. Note that you do not need to know about inverse hyperbolic functions to use the formulas on this page.

u–Substitution

Often, an integrand will contain a function within a function. For example, in the integral $\int \frac{ln\sqrt{x}}{x}\,dx$, we have the function $\sqrt{x}$ within the ln function. When this happens, it is often useful to substitute another variable for the internal function. Typically the variable u is used to represent the inner function, so the process is called **u–substitution.**

The typical process used for u–substitution is described in steps below. When trying this approach, note the following:

- u–substitution will work for all integrals, even ones that look ripe for it, though it does work often.
- If one attempted substitution does not work, the student should try another one. It takes practice to train the eye to identify what functions work well in this process.
- It is possible that the student will be faced with an integral than simply cannot be integrated by any elementary method (e.g., $\int e^{-x^2}\,dx$).

Process

Following are the steps for the general solution to an integral using u–substitution.

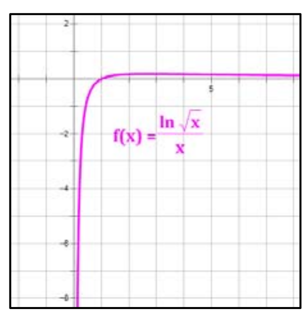
1. Set a portion of the integrand equal to a new variable, e.g., u. Look to the rest of the integrand in deciding what to set equal to u. You will need to have du in the integrand as well, if this technique is to find success.
2. Find du in terms of dx.
3. Rearrange the integrand so that the integral exists in terms of u instead of x.
4. Perform the integration.
5. Substitute the expression for u back into the result of the integration.
6. If you are uncomfortable with the result, integrate it to see if you get the integrand as a result. If so, you have achieved your goal. And, don't forget the $+C$ for an indefinite integration.

Example 5.1: Find: $\int \frac{\ln\sqrt{x}}{x}\,dx$

$$\int \frac{\ln\sqrt{x}}{x}\,dx = \int \frac{\frac{1}{2}\ln x}{x}\,dx = \frac{1}{2}\int \ln x\,\frac{1}{x}\,dx$$

$$= \frac{1}{2}\int u\,du = \frac{1}{2}\cdot\frac{1}{2}\,u^2 = \frac{1}{4}\,(\ln x)^2 + C$$

$$\boxed{\begin{aligned} u &= \ln x \\ du &= \frac{1}{x}\,dx \end{aligned}}$$



Example 5.2: Find: $\int \frac{(1-\ln\ t)^2}{t}\, dt$

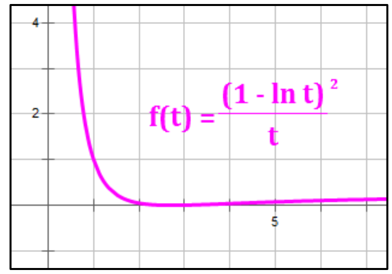
$$\int \frac{(1-\ln\ t)^2}{t}\, dt$$

$$= -\int (1-\ln\ t)^2 \left(-\frac{1}{t}\, dt\right)$$

$$= -\int u^2\, du = -\frac{1}{3} u^3 = -\frac{1}{3}(1-\ln t)^3 + C$$

$u = 1 - \ln t$

$du = -\frac{1}{t}\, dt$



Example 5.3: Find: $\int \frac{dx}{\sqrt{9-x^2}}$ Recall: $\int \frac{1}{\sqrt{1-u^2}}\, du = \sin^{-1} u$ with $-\frac{\pi}{2} \le \sin^{-1} u \le \frac{\pi}{2}$

$$\int \frac{dx}{\sqrt{9-x^2}}$$

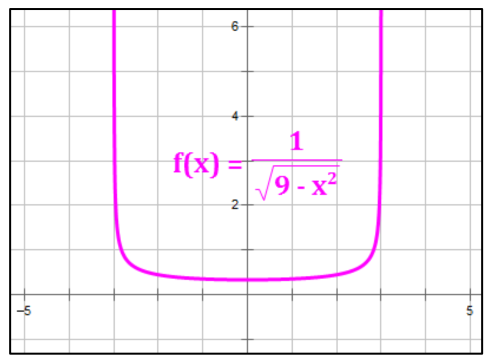
$$= \int \frac{1}{3\sqrt{1-\left(\frac{x}{3}\right)^2}}\, dx$$

$$= \int \frac{1}{\sqrt{1-\left(\frac{x}{3}\right)^2}}\, \frac{1}{3}\, dx$$

$$= \sin^{-1} u + C = \sin^{-1}\left(\frac{x}{3}\right) + C$$

$u = \frac{x}{3}$

$du = \frac{1}{3} dx$



Example 5.4: Find: $\int \frac{e^x}{1+e^{2x}}\, dx$ Recall: $\int \frac{1}{1+u^2}\, du = \tan^{-1} u$ with $-\frac{\pi}{2} \le \tan^{-1} u \le \frac{\pi}{2}$

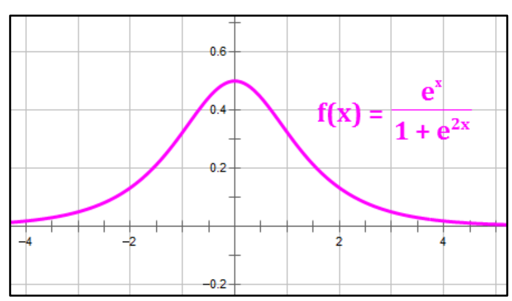
$$\int \frac{e^x}{1+e^{2x}}\, dx$$

$$= \int \frac{1}{1+(e^x)^2}\, e^x dx$$

$$= \int \frac{1}{1+u^2}\, du = \tan^{-1} u = \tan^{-1}(e^x) + C$$

$u = e^x$

$du = e^x\, dx$



Partial Fractions

Partial Fractions

Every rational function of the form $R(x) = \dfrac{N(x)}{D(x)}$ can be expressed as a sum of fractions with linear and quadratic forms in their denominators. For example:

$$\frac{x^4 + 2x^3 - 3x + 4}{(x-4)^3(x^2+2x+4)^2} = \frac{a_1}{(x-4)} + \frac{a_2}{(x-4)^2} + \frac{a_3}{(x-4)^3} + \frac{b_1x + c_1}{(x^2+2x+4)} + \frac{b_2x + c_2}{(x^2+2x+4)^2}$$

Our task is to determine the appropriate fractions, including the values of the a's, b's and c's, so we can integrate the function. The result of integration tends to contain a number of natural logarithm terms and inverse tangent terms, as well as others.

The following process can be used to determine the set of fractions (including the a's, b's and c's) whose sum is equal to $R(x)$.

Process

1. If $N(x)$ has the same degree or higher degree than $D(x)$, divide $N(x)$ by $D(x)$ to obtain the non-fractional (polynomial) component of the rational function. Proceed in the next steps with the fractional component of the rational function.

   Example 5.5: $R(x) = \frac{x^2+2x-5}{x-2} = x + 4 + \frac{3}{x-2}$.

   Since it is easy to integrate the polynomial portion of this result, (i.e., $x + 4$), it remains to integrate the fractional portion (i.e., $\frac{3}{x-2}$).

2. To determine the denominators of the fractions on the right side of the equal sign, we must first factor the denominator of $R(x)$, i.e., $D(x)$.

   Note that every polynomial can be expressed as the product of linear terms and quadratic terms, so that:

   $$D(x) = k(x - r_1)(x - r_2) \ldots (x - r_n) \cdot (x^2 + p_1x + q_1)(x^2 + p_2x + q_2) \ldots (x^2 + p_mx + q_m)$$

   Where k is the lead coefficient, the $(x - r_i)$ terms are the linear factors and the $(x^2 + p_ix + q_i)$ are the quadratic terms of $D(x)$.

3. Every rational function can be expressed as the sum of fractions of the following types:

$$\frac{a_i}{(x-r_i)^s} \quad \text{or} \quad \frac{b_i x+c_i}{(x^2+p_i x+q_i)^t}$$

Where the exponents in the denominators, s and t, take all values from 1 to the multiplicity of the factor in $D(x)$.

Examples 5.6 – 5.8:

$$\frac{2x^2+5x-3}{(x+2)^3} = \frac{a_1}{(x+2)} + \frac{a_2}{(x+2)^2} + \frac{a_3}{(x+2)^3}$$

$$\frac{x^3-x^2+6x-2}{(x^2-3x+7)^2} = \frac{b_1 x+c_1}{(x^2-3x+7)} + \frac{b_2 x+c_2}{(x^2-3x+7)^2}$$

$$\frac{x^4+2x^3-3x+4}{(x-1)^2(x+3)(x^2-4x+1)} = \frac{a_1}{(x-1)} + \frac{a_2}{(x-1)^2} + \frac{a_3}{(x+3)} + \frac{b_1 x+c_1}{(x^2-4x+1)}$$

We must solve for the values of the a's, b's and c's. This is accomplished by obtaining a common denominator and then equating the coefficients of each term in the numerator. This will generate a number of equations with the same number of unknown values of a, b and c.

Example 5.6a (using the first expression above):

$$\frac{2x^2+5x-3}{(x+2)^3} = \frac{a_1}{(x+2)} + \frac{a_2}{(x+2)^2} + \frac{a_3}{(x+2)^3}$$

$$= \frac{a_1(x+2)^2}{(x+2)^3} + \frac{a_2(x+2)}{(x+2)^3} + \frac{a_3}{(x+2)^3} = \frac{a_1(x+2)^2 + a_2(x+2) + a_3}{(x+2)^3}$$

Equating the numerators, then,

$$2x^2+5x-3 \;=\; a_1 x^2 + (4a_1+a_2)x + (4a_1+2a_2+a_3)$$

So that:

$$\left.\begin{aligned} a_1 &= 2 \\ 4a_1+a_2 &= 5 \\ 4a_1+2a_2+a_3 &= -3 \end{aligned}\right\} \quad \xrightarrow{\text{We solve these equations to obtain:}} \quad \left\{\begin{aligned} a_1 &= 2 \\ a_2 &= -3 \\ a_3 &= -5 \end{aligned}\right.$$

Finally concluding that:

$$\frac{2x^2+5x-3}{(x+2)^3} \;=\; \frac{2}{(x+2)} + \frac{-3}{(x+2)^2} + \frac{-5}{(x+2)^3} \;=\; \frac{2}{(x+2)} - \frac{3}{(x+2)^2} - \frac{5}{(x+2)^3}$$

4. The final step is to integrate the resulting fractions.

Example 5.6b (continuing from Step 3):

$$\int \frac{2x^2 + 5x - 3}{(x+2)^3}\,dx = \int \frac{2}{(x+2)} - \frac{3}{(x+2)^2} - \frac{5}{(x+2)^3}\,dx$$

$$= 2\ln|x+2| + \frac{3}{(x+2)} + \frac{5}{2(x+2)^2}$$

Integration by Parts

General

From the product rule of derivatives we have:

$$d\,uv \;=\; u\,dv + v\,du$$

Rearranging terms we get:

$$u\,dv \;=\; d\,uv \;-\; v\,du$$

Finally, integrating both sides gives us:

$$\int u\,dv \;=\; \int d\,uv \;-\; \int v\,du$$

$$\int u\,dv \;=\; uv \;-\; \int v\,du$$

This last formula is the one for integration by parts and is extremely useful in solving integrals.

When performing an integration by parts, first define u and dv.

L I A T E

When integrating by parts, students often struggle with how to break up the original integrand into u and dv. LIATE is an acronym that is often used to determine which part of the integrand should become u. Here's how it works: let u be the function from the original integrand that shows up first on the list below.

- Logarithmic functions (e.g., $\ln x$)
- Inverse trigonometric functions (e.g., $\tan^{-1} x$)
- Algebraic functions (e.g., $x^3 + x - 2$)
- Trigonometric functions (e.g., $\cos x$)
- Exponential functions (e.g., e^x)

In general, we want to let u be a function whose derivative du is both relatively simple and compatible with v. Logarithmic and inverse trigonometric functions appear first in the list because their derivatives are algebraic; so if v is algebraic, $v\,du$ is algebraic and an integration with "weird" functions is transformed into one that is completely algebraic. Note that the LIATE approach does not always work, but in many cases it can be helpful.

Example 5.9: Find $\int \cos^2 x\, dx$ (note: ignore the $+C$ until the end)

$$\int \cos^2 x\, dx = \sin x \cos x - \int (-\sin^2 x)\, dx$$

$$= \sin x \cos x + \int (\sin^2 x)\, dx$$

$$= \sin x \cos x + \int (1 - \cos^2 x)\, dx$$

$$= \sin x \cos x + \int (1)\, dx - \int (\cos^2 x)\, dx$$

$$\int \cos^2 x\, dx = \sin x \cos x + x - \int (\cos^2 x)\, dx$$

$$2 \int \cos^2 x\, dx = \sin x \cos x + x$$

$$\int \cos^2 x\, dx = \frac{1}{2}(\sin x \cos x + x) + C$$

Let:

$u = \cos x$	$v = \sin x$
$du = -\sin x\, dx$	$dv = \cos x\, dx$

Example 5.9A: Find $\int \cos^2 x\, dx$ without using integration by parts

Let's use the Trig identity: $\cos^2 x = \frac{1+\cos 2x}{2}$

$$\int \cos^2 x\, dx = \int \left(\frac{1 + \cos 2x}{2}\right) dx$$

$$= \frac{1}{2} \int (1 + \cos 2x)\, dx$$

$$= \frac{1}{2}\left(x + \frac{1}{2} \sin 2x\right) + C$$

Next, recall that $\sin 2x = 2 \sin x \cos x$. So,

$$\int \cos^2 x\, dx = \frac{1}{2}\left(x + \frac{1}{2} \cdot 2 \sin x \cos x\right) + C$$

$$= \frac{1}{2}(\sin x \cos x + x) + C$$

Example 5.10: Find $\int \ln x \, dx$

$$\int \ln x \, dx = x \ln x - \int \frac{x}{x} dx$$

$$= x \ln x - \int 1 \, dx$$

$$= x \ln x - x + C$$

Let: $u = \ln x$ $\quad v = x$

$du = \frac{1}{x} dx$ $\quad dv = dx$

Example 5.11: Find $\int (x^2 e^x) \, dx$

$$\int (x^2 e^x) \, dx = x^2 e^x - \int 2x \, e^x \, dx$$

$$= x^2 e^x - 2 \int x \, e^x \, dx$$

$$= x^2 e^x - 2 \left(x \, e^x - \int e^x \, dx \right)$$

$$= x^2 e^x - 2(x \, e^x - e^x) + C$$

$$= (x^2 - 2x + 2)e^x + C$$

Let: $u = x^2$ $\quad v = e^x$

$du = 2x \, dx$ $\quad dv = e^x \, dx$

Let: $u = x$ $\quad v = e^x$

$du = dx$ $\quad dv = e^x \, dx$

Example 5.12: Find $\int \tan^{-1} x \, dx$

$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \int \frac{x}{1 + x^2} dx$$

$$= x \tan^{-1} x - \frac{1}{2} \int \frac{1}{1 + x^2} \, 2x \, dx$$

$$= x \tan^{-1} x - \frac{1}{2} \ln(1 + x^2) + C$$

Let: $u = \tan^{-1} x$ $\quad v = x$

$du = \frac{1}{1 + x^2} dx$ $\quad dv = dx$

Integration by Parts – Tabular Method

If the use of Integration by Parts results in another integral that must be solved using integration by parts, the **Tabular Method** can be used to simplify repeating the process and save time. This method is particularly useful when one of the terms of the integrand is a polynomial.

Description of the Method

Create a table like the one below, starting with the u- and dv-substitutions to be used in the initial integration by parts. Start the dv-column one line higher than the u-column.

- In the u-column, take consecutive derivatives until the derivative equals zero.
- In the dv-column, take consecutive integrals until the derivative column equals zero.
- In the sign column, begin with a $+$ sign and alternate $+$ and – signs.
- Multiply the sign and the expressions in the u- and dv columns to obtain each term of the solution.
- Add all of the terms obtained as described above to obtain the complete solution.

Example 5.13: Tabular Method to determine $\int x^3 \sin 2x\, dx$

Terms	Sign	$u, du, d^2u \ldots$		$dv, v, \int v, \iint v, \ldots$	
dv				$\sin 2x\, dx$	
u, v	$+$	x^3	take consecutive derivatives	$-\frac{1}{2}\cos 2x$	take consecutive integrals
$du, \int v$	$-$	$3x^2$		$-\frac{1}{4}\sin 2x$	
$d^2u, \iint v$	$+$	$6x$		$\frac{1}{8}\cos 2x$	
$d^3u, \iiint v$	$-$	6		$\frac{1}{16}\sin 2x$	

Solution:

$$\int x^3 \sin 2x\, dx = (x^3)\left(-\frac{1}{2}\cos 2x\right) - (3x^2)\left(-\frac{1}{4}\sin 2x\right) + (6x)\left(\frac{1}{8}\cos 2x\right) - 6\left(\frac{1}{16}\sin 2x\right) + C$$

$$= -\frac{1}{2}x^3\cos 2x + \frac{3x^2}{4}\sin 2x + \frac{3x}{4}\cos 2x - \frac{3}{8}\sin 2x + C$$

Trigonometric Substitution

Certain integrands are best handled with a trigonometric substitution. Three common forms are shown in the table below:

Integral Contains this Form	Try this Substitution
$\sqrt{x^2+a^2}$ *or* $\sqrt{a^2+x^2}$	$x=a\tan\theta$
$\sqrt{x^2-a^2}$	$x=a\sec\theta$
$\sqrt{a^2-x^2}$	$x=a\sin\theta$ or $x=a\cos\theta$

Why are these helpful? Quite simply because they eliminate what is often the most difficult part of the problem – the square root sign. Let's look at each of the substitutions in the table.

- Using the substitution $x=a\tan\theta,$ we have:

  $$\sqrt{x^2+a^2}=\sqrt{(a\tan\theta)^2+a^2}=\sqrt{a^2\,(\tan^2\theta+1)}=\sqrt{a^2\,\sec^2\theta}=a\sec\theta$$

- Using the substitution $x=a\sec\theta,$ we have:

  $$\sqrt{x^2-a^2}=\sqrt{(a\sec\theta)^2-a^2}=\sqrt{a^2\,(\sec^2\theta-1)}=\sqrt{a^2\,\tan^2\theta}=a\tan\theta$$

- Using the substitution $x=a\sin\theta,$ we have:

  $$\sqrt{a^2-x^2}=\sqrt{a^2-(a\sin\theta)^2}=\sqrt{a^2\,(1-\sin^2\theta)}=\sqrt{a^2\,\cos^2\theta}=a\cos\theta$$

- Using the substitution $x=a\cos\theta,$ we have:

  $$\sqrt{a^2-x^2}=\sqrt{a^2-(a\cos\theta)^2}=\sqrt{a^2\,(1-\cos^2\theta)}=\sqrt{a^2\,\sin^2\theta}=a\sin\theta$$

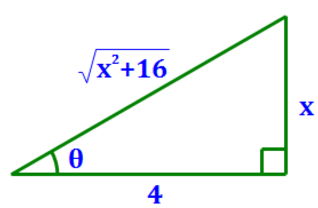
Example 5.14:

$$\int\frac{dx}{x\,\sqrt{x^2+16}}$$

Let: $x=4\tan\theta$

$dx=4\sec^2\theta\,d\theta$

$$=\int\frac{4\sec^2\theta\,d\theta}{4\tan\theta\,\sqrt{(4\tan\theta)^2+16}}$$



$$=\int\frac{4\sec^2\theta\,d\theta}{4\tan\theta\cdot 4\sec\theta}$$

$$=\frac{1}{4}\int\frac{\sec\theta}{\tan\theta}d\theta=\frac{1}{4}\int\csc\theta\,d\theta$$

$$=-\frac{1}{4}\ln|\csc\theta+\cot\theta|+C\;=-\frac{1}{4}\ln\left|\frac{\sqrt{x^2+16}+\;4}{x}\right|+C$$

Example 5.15:

$$\int_0^1 \sqrt{x^2+1}\,dx$$

Let: $x = \tan\theta$, so $\theta = \tan^{-1} x$ and $dx = \sec^2\theta\, d\theta$

Then the limits of integration become: $\theta = \tan^{-1} 0 = 0$ and $\theta = \tan^{-1} 1 = \frac{\pi}{4}$.

$$\int_0^1 \sqrt{x^2+1}\,dx = \int_0^{\frac{\pi}{4}} \sqrt{\tan^2\theta+1}\cdot \sec^2\theta\,d\theta = \int_0^{\frac{\pi}{4}} \sqrt{\sec^2\theta}\cdot \sec^2\theta\,d\theta = \int_0^{\frac{\pi}{4}} \sec^3\theta\,d\theta$$

We need integration by parts to integrate $(\sec^3\theta\,d\theta)$.

Let:	
$u = \sec\theta$	$v = \tan\theta$
$du = \sec\theta\tan\theta\,d\theta$	$dv = \sec^2\theta\,d\theta$

$$\int_0^{\frac{\pi}{4}} \sec^3\theta\,d\theta = \sec\theta\tan\theta\Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec\theta\tan^2\theta\,d\theta$$

$$\int_0^{\frac{\pi}{4}} \sec^3\theta\,d\theta = \sec\theta\tan\theta\Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec\theta\,(\sec^2\theta - 1)d\theta$$

$$\int_0^{\frac{\pi}{4}} \sec^3\theta\,d\theta = \sec\theta\tan\theta\Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec^3\theta\,d\theta + \int_0^{\frac{\pi}{4}} \sec\theta\,d\theta$$

Next, notice that $\left(\int_0^{\frac{\pi}{4}} \sec^3\theta\,d\theta\right)$ is on both sides of the equation. So, add it to both sides:

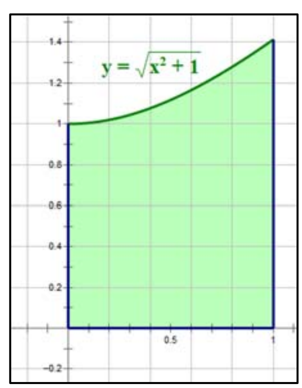
$$2\int_0^{\frac{\pi}{4}} \sec^3\theta\,d\theta = \sec\theta\tan\theta\Big|_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \sec\theta\,d\theta$$

$$2\int_0^{\frac{\pi}{4}} \sec^3\theta\,d\theta = \sec\theta\tan\theta\Big|_0^{\frac{\pi}{4}} + \ln|\sec\theta+\tan\theta|\Big|_0^{\frac{\pi}{4}}$$

$$2\int_0^{\frac{\pi}{4}} \sec^3\theta\,d\theta = \left(\sec\frac{\pi}{4}\tan\frac{\pi}{4} - \sec 0\tan 0\right) + \left(\ln\left|\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right| - \ln|\sec 0+\tan 0|\right)$$

$$2\int_0^{\frac{\pi}{4}} \sec^3\theta\,d\theta = (\sqrt{2}\cdot 1 - 1\cdot 0) + \left[\,\ln(\sqrt{2}+1) - \ln(1+0)\,\right]$$

$$2\int_0^{\frac{\pi}{4}} \sec^3\theta\,d\theta = \sqrt{2} + \ln(\sqrt{2}+1)$$



Finally, divide both sides by 2:

$$\int_0^1 \sqrt{x^2+1}\,dx = \int_0^{\frac{\pi}{4}} \sec^3\theta\,d\theta = \frac{\sqrt{2}+\ln(\sqrt{2}+1)}{2} \sim 1.14779$$

Gamma Function

The **Gamma Function** is defined by the following definite integral:

$$\Gamma(x) = \int_0^\infty (t^{x-1}e^{-t})\,dt$$

In this context, x is a constant and t is the variable in the integrand. Using integration by parts:

Let: $u = t^x$	$v = -e^{-t}$
$du = x\,t^{x-1}\,dt$	$dv = e^{-t}\,dt$

$$\begin{aligned}\Gamma(x+1) &= \int_0^\infty (t^x e^{-t})\,dt \\ &= -t^x e^{-t}\Big|_0^\infty - \int_0^\infty (-x\,t^{x-1}e^{-t})\,dt \\ &= \left[\lim_{t\to\infty}(-t^x e^{-t}) - 0\right] + x\int_0^\infty (\,t^{x-1}e^{-t})\,dt \\ &= [0] + x\,\Gamma(x)\end{aligned}$$

So, we obtain one of the key properties of the Gamma Function:

$$\boxed{\boldsymbol{\Gamma(x+1) = x\,\Gamma(x)}}$$

Next, let's compute: $\Gamma(1) = \int_0^\infty (e^{-t})\,dt$

$$\Gamma(1) = \int_0^\infty (e^{-t})\,dt = -e^{-t}\Big|_0^\infty = -(0-1)$$

$$\boxed{\boldsymbol{\Gamma(1) = 1}}$$

Now for something especially cool. Based on these two results, we have the following:

$$\begin{aligned}\Gamma(1) &= 1 \\ \Gamma(2) &= 1\cdot\Gamma(1) = 1\cdot 1 = 1 = 1! \\ \Gamma(3) &= 2\cdot\Gamma(2) = 2\cdot 1 = 2 = 2! \\ \Gamma(4) &= 3\cdot\Gamma(3) = 3\cdot 2 = 6 = 3! \\ \Gamma(5) &= 4\cdot\Gamma(4) = 4\cdot 6 = 24 = 4! \\ &\dots\end{aligned}$$

$$\boxed{\boldsymbol{\Gamma(x+1) = x!}}$$

Using the Gamma Function to Solve a Definite Integral

Example 5.16:

$$\int_0^\infty e^{-x^3} dx$$

Let: $t = x^3$, so $x = t^{1/3}$, $dx = \frac{1}{3} t^{-2/3}\, dt$

The limits of integration do not change.

Then:

$$\int_0^\infty e^{-x^3} dx = \int_0^\infty e^{-t} \cdot \frac{1}{3} t^{-2/3}\, dt$$

$$= \frac{1}{3} \int_0^\infty t^{-2/3}\, e^{-t} dt$$

Compare this to the Gamma Function:

$$\Gamma(x) = \int_0^\infty t^{x-1}\, e^{-t} dt$$

If: $x = \frac{1}{3}$, we get:

$$\int_0^\infty e^{-x^3} dx = \frac{1}{3} \int_0^\infty t^{-2/3}\, e^{-t} dt = \frac{1}{3} \int_0^\infty t^{(1/3-1)}\, e^{-t} dt$$

$$= \frac{1}{3} \Gamma\left(\frac{1}{3}\right) = \boldsymbol{\Gamma\left(\frac{4}{3}\right)} \sim \mathbf{0.89298}$$

The following properties of the Gamma Function are of particular interest:

- $\boldsymbol{\Gamma(x+1) = x\,!}$ for integer values of x
- $\boldsymbol{\Gamma(x+1) = x \cdot \Gamma(x)}$ for values of x where $\Gamma(x)$ exists
- $\boldsymbol{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$
- $\boldsymbol{\Gamma(x) \cdot \Gamma(1-x) = \dfrac{\pi}{\sin(\pi x)}}$ for $\{0 < x < 1\}$
- $\boldsymbol{\Gamma(x) \neq 0}$ for any value of x
- $\boldsymbol{\Gamma(x)}$ does not exist for $x = 0$, nor for negative integer values of x.

Some values of $\boldsymbol{\Gamma(x)}$:

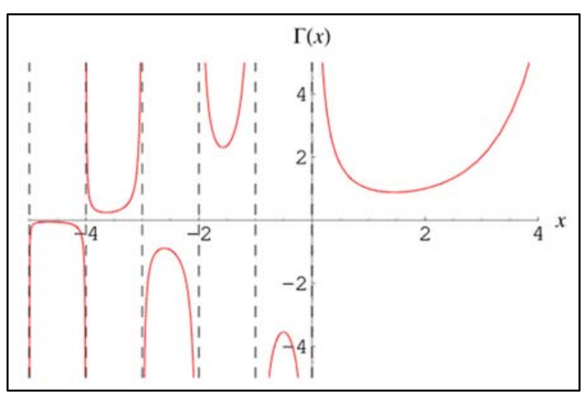
$\Gamma(1) = 0$ $\quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$\Gamma(2) = 1$ $\quad \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$

$\Gamma(3) = 2$ $\quad \Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4}$

$\Gamma(4) = 6$ $\quad \Gamma\left(\frac{7}{2}\right) = \frac{15\sqrt{\pi}}{8}$

$\Gamma\left(\frac{1}{3}\right) \sim 2.67894$



Graph from: mathworld.wolfram.com/GammaFunction.html

Note: There are a number of online Gamma Function calculators that provide values of $\Gamma(x)$.

Beta Function

The Beta Function is defined by the following equivalent definite integrals:

$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}\,(1-t)^{y-1}dt$$

$$\mathrm{B}(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}dt$$

Relation between Beta and Gamma Function

$\mathrm{B}(x,y) = \mathrm{B}(y,x)$ That is, the Beta function is symmetric.

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \frac{(x-1)!\,(y-1)!}{(x+y-1)!} \longleftarrow \textbf{Very useful!}$$

Note: mathworld.wolfram.com/BetaFunction.html lists many other properties of the Beta Function.

Example 5.17:

$$\int_0^1 t^4\,(1-t)^6 dt = \mathrm{B}(5,7) = \frac{\Gamma(5)\Gamma(7)}{\Gamma(5+7)} = \frac{4!\cdot 6!}{11!} = \mathbf{\frac{1}{2310}}$$

Trigonometric Form

Rewrite the Beta Function integral with dummy variable u instead of t.

$$\mathrm{B}(x,y) = \int_0^1 u^{x-1}\,(1-u)^{y-1}du$$

Substitute: $u = \sin^2 t$, so $du = 2\sin t\cos t\,dt$, $u\big|_0^1 \Rightarrow t\,\Big|_0^{\pi/2}$ to get:

$$\mathrm{B}(x,y) = 2\int_0^{\pi/2} \sin^{2x-1}(t)\cos^{2y-1}(t)\,dt \longleftarrow \textbf{Very useful!}$$

Example 5.18:

$$\int_0^{\pi/2} \sin^9(t)\cos^5(t)\,dt = \frac{1}{2}\mathrm{B}(5,3) = \frac{1}{2}\cdot\frac{\Gamma(5)\Gamma(3)}{\Gamma(5+3)} = \frac{4!\cdot 2!}{2\cdot 7!} = \mathbf{\frac{1}{210}}$$

Note, in this example: $2x-1=9$, so $x=5$, and $2y-1=5$, so $y=3$.

Example 5.19: Find the value of: $\int_0^{\pi/2} \frac{\sqrt{\tan x}}{(\cos x + \sin x)^2} dx$

$$\int_0^{\pi/2} \frac{\sqrt{\tan x}}{(\cos x + \sin x)^2} dx = \int_0^{\pi/2} \frac{\sqrt{\tan x}}{(\cos^2 x)(1+\tan x)^2} dx$$

Let: $t = \tan x$, so $dt = \sec^2 x = \frac{1}{\cos^2 x} dx$, $x \Big|_0^{\pi/2} \Rightarrow t \Big|_0^{\infty}$

$$\int_0^{\pi/2} \frac{\sqrt{\tan x}}{(\cos x + \sin x)^2} dx = \int_0^{\infty} \frac{t^{1/2}}{(1+t)^2} dt$$

This integral is now in the second Beta Function form shown above:

$$\mathrm{B}(x,y) = \int_0^{\infty} \frac{t^{x-1}}{(1+t)^{x+y}} dt, \quad x - 1 = \frac{1}{2}, \quad x + y = 2 \quad \Rightarrow \quad x = \frac{3}{2}, \quad y = \frac{1}{2}$$

$$\int_0^{\pi/2} \frac{\sqrt{\tan x}}{(\cos x + \sin x)^2} dx = \int_0^{\infty} \frac{t^{(3/2-1)}}{(1+t)^{(3/2+1/2)}} dt$$

$$= \mathrm{B}\left(\frac{3}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{1}{2}\right)} = \frac{\frac{\sqrt{\pi}}{2} \cdot \sqrt{\pi}}{1!} = \boldsymbol{\frac{\pi}{2}}$$

Impossible Integrals

Some expressions are impossible to integrate using elementary methods. Examples are provided below.

Error Function

$\int e^{-x^2}\,dx$ This integral may be encountered in exercises related to the normal probability distribution. It is important enough that tables of value associated with its definite integral form have been developed.

$$\operatorname{erf}(x) = \int_0^x e^{-t^2}\,dt$$

Other Functions with Tables of Values

A number of other integrals are important enough to have tables of values developed for them:

Function Name	Indefinite Form	Definite Form
Logarithmic Integral	$\int \frac{1}{\ln x}\,dx$	$li(x) = \int_0^x \frac{1}{\ln t}\,dt$
Sine Integral	$\int \frac{\sin x}{x}\,dx$	$Si(x) = \int_0^x \frac{\sin t}{t}\,dt$
Cosine Integral	$\int \frac{\cos x}{x}\,dx$	$Ci(x) = -\int_x^\infty \frac{\cos t}{t}\,dt$
Exponential Integral	$\int \frac{e^{-x}}{x}\,dx$	$Ei(x) = -\int_{-x}^\infty \frac{e^{-t}}{t}\,dt$

Other Impossible Integrals

$$\int \sin\left(\frac{1}{x}\right)dx,\quad \int \cos\left(\frac{1}{x}\right)dx,\quad \int \tan\left(\frac{1}{x}\right)dx,\quad \int \sin\left(\sqrt{x}\right)dx,\quad \int \sqrt{x}\sin x\,dx,\quad \int \sin(x^2)\,dx$$

$$\int e^{(x^2)}dx,\quad \int e^{(1/x)}dx,\quad \int \frac{e^x}{x}dx,\quad \int \ln(\ln x)\,dx,\quad \int \ln(\sin x)\,dx,\quad \int \frac{x}{\ln x}dx$$

Many more functions that cannot be integrated using elementary methods can be found at:

https://owlcation.com/stem/List-of-Functions-You-Cannot-Integrate-No-Antiderivatives

Hyperbolic Functions

Definitions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$

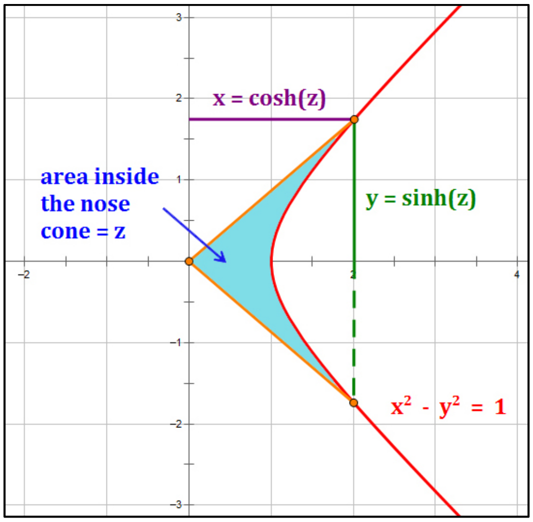
$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} \qquad \operatorname{csch} x = \frac{2}{e^x - e^{-x}}$$

Geometric Representation

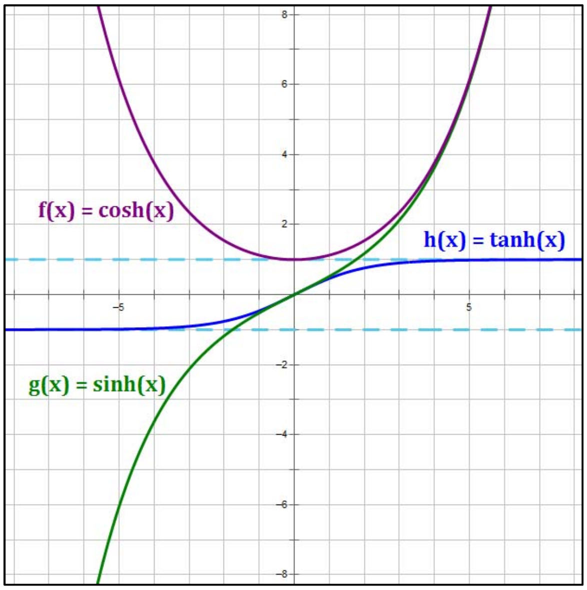
The illustration at right provides a geometric representation of a value "z" and its hyperbolic function values relative to the unit hyperbola.

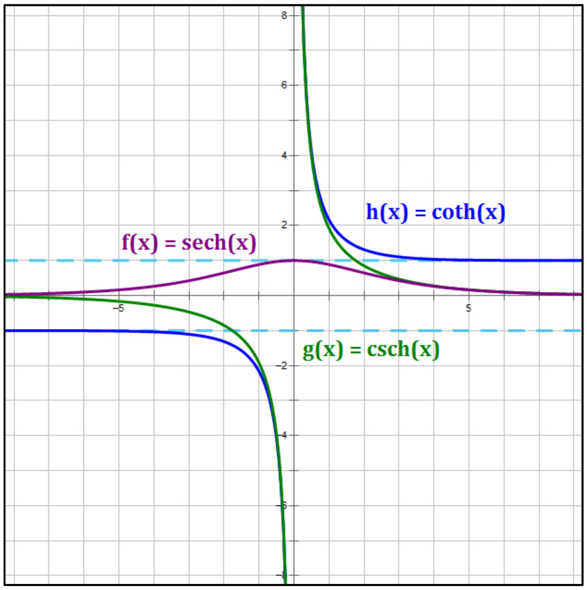
The hyperbolic cosine "$y = \cosh(x)$", is the equation of the Catenary, the shape of hanging chain that is supported at both ends.

Many of the properties of hyperbolic functions bear a striking resemblance to the corresponding properties of trigonometric functions (see next page).



Graphs of Hyperbolic Functions





Hyperbolic Function Identities

Comparison of Trigonometric and Hyperbolic Identities

Hyperbolic Function Identity	Trigonometric Function Identity
$\sinh(-x) = -\sinh x$	$\sin(-x) = -\sin x$
$\cosh(-x) = \cosh x$	$\cos(-x) = \cos x$
$\tanh(-x) = -\tanh x$	$\tan(-x) = -\tan x$

$\cosh^2 x - \sinh^2 x = 1$	$\sin^2 x + \cos^2 x = 1$
$\operatorname{sech}^2 x = 1 - \tanh^2 x$	$\sec^2 x = 1 + \tan^2 x$
$\operatorname{csch}^2 x = \coth^2 x - 1$	$\csc^2 x = 1 + \cot^2 x$

$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$	$\sin(x+y) = \sin x \cos y + \cos x \sin y$
$\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$	$\sin(x-y) = \sin x \cos y - \cos x \sin y$
$\sinh 2x = 2 \sinh x \cosh x$	$\sin 2x = 2 \sin x \cos x$
$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$	$\cos(x+y) = \cos x \cos y - \sin x \sin y$
$\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$	$\cos(x-y) = \cos x \cos y + \sin x \sin y$
$\cosh 2x = \cosh^2 x + \sinh^2 x$	$\cos 2x = \cos^2 x - \sin^2 x$
$\tanh(x+y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$	$\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$
$\tanh(x-y) = \dfrac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$	$\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$

$\sinh^2 x = \dfrac{-1 + \cosh 2x}{2}$	$\sin^2 x = \dfrac{1 - \cos 2x}{2}$
$\cosh^2 x = \dfrac{1 + \cosh 2x}{2}$	$\cos^2 x = \dfrac{1 + \cos 2x}{2}$

Hyperbolic Function Identities

Relationship between Trigonometric and Hyperbolic Functions

$$\left.\begin{aligned}\sinh x &= -i\sin(ix)\\ \cosh x &= \cos(ix)\end{aligned}\right\}$$ From these two relationships, the other four may be determined.

$$\tanh x = \frac{\sinh x}{\cosh x} = -i\tan(ix)$$

$$\coth x = \frac{\cosh x}{\sinh x} = i\cot(ix)$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \sec(ix)$$

$$\operatorname{csch} x = \frac{1}{\sinh x} = i\csc(ix)$$

Series Expansions

Appendix G provides series expansions for the trigonometric functions $\sin x$ and $\cos x$. Those are repeated here, along with the series expansions for the corresponding hyperbolic functions $\sinh x$ and $\cosh x$.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \qquad \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

It is possible to develop series expansions for the other four hyperbolic functions, but they involve the more esoteric Bernoulli numbers and Euler numbers. Instead, the student may wish to develop values for the other four hyperbolic functions from the expansions of $\sinh x$ and $\cosh x$.

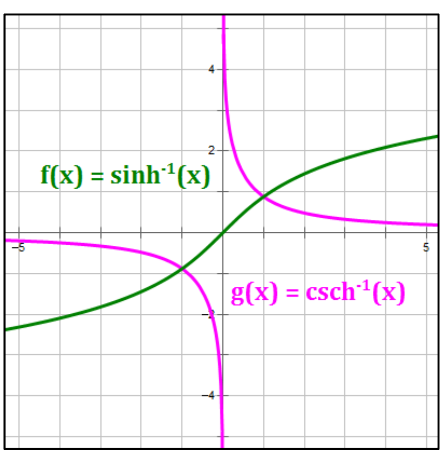
Example 6.1: $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots}{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots}$

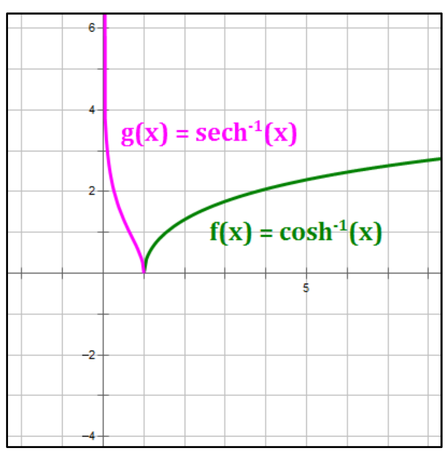
Inverse Hyperbolic Functions

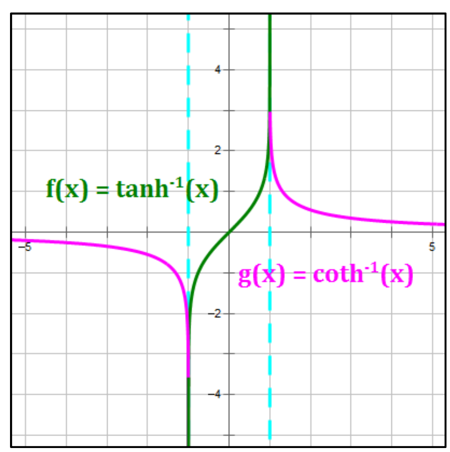
Logarithmic Forms of Inverse Hyperbolic Functions

Principal Values	Function Domain	Function Range
$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$	$(-\infty, \infty)$	$(-\infty, \infty)$
$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$	$[1, \infty)$	$[0, \infty)$
$\tanh^{-1} x = \dfrac{1}{2} \ln\left(\dfrac{1+x}{1-x}\right)$	$(-1, 1)$	$(-\infty, \infty)$
$\coth^{-1} x = \tanh^{-1}\left(\dfrac{1}{x}\right) = \dfrac{1}{2} \ln\left(\dfrac{x+1}{x-1}\right)$	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, \infty)$
$\operatorname{sech}^{-1} x = \cosh^{-1}\left(\dfrac{1}{x}\right) = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right)$	$(0, 1]$	$[0, \infty)$
$\operatorname{csch}^{-1} x = \sinh^{-1}\left(\dfrac{1}{x}\right) = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{1+x^2}}{\lvert x\rvert}\right)$	$(-\infty, \infty)$	$(-\infty, \infty)$

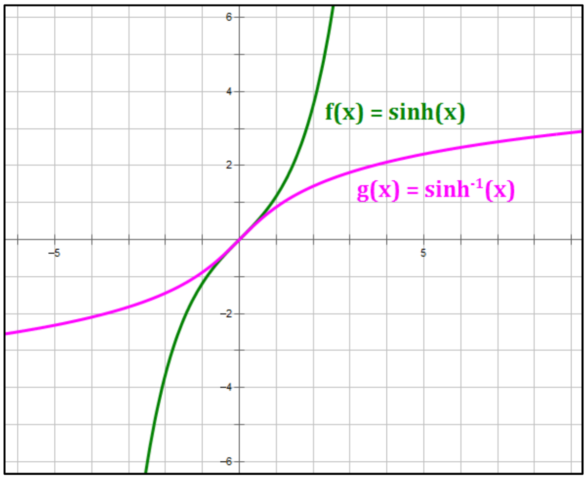
Graphs of Inverse Hyperbolic Functions

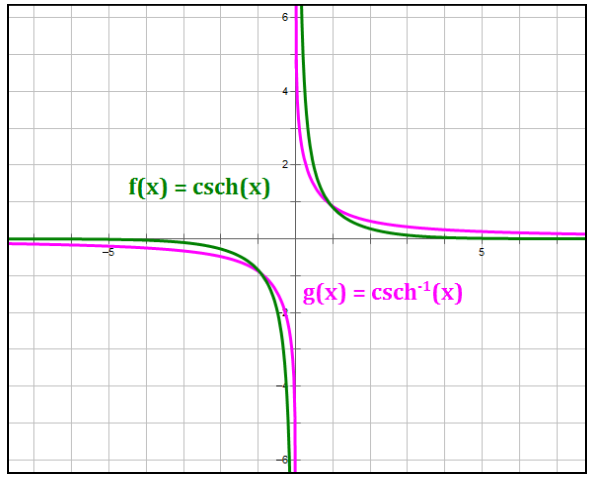


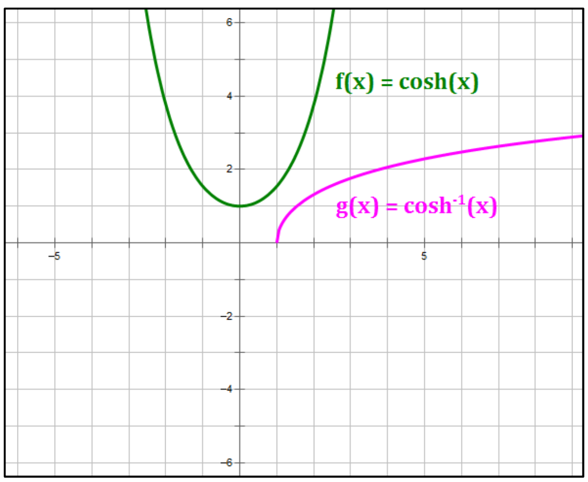


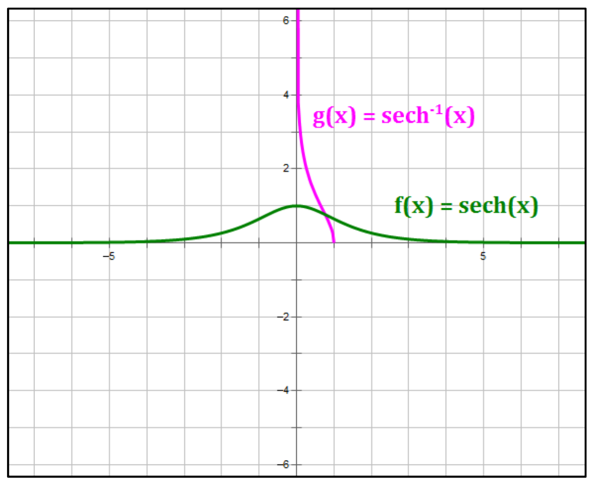


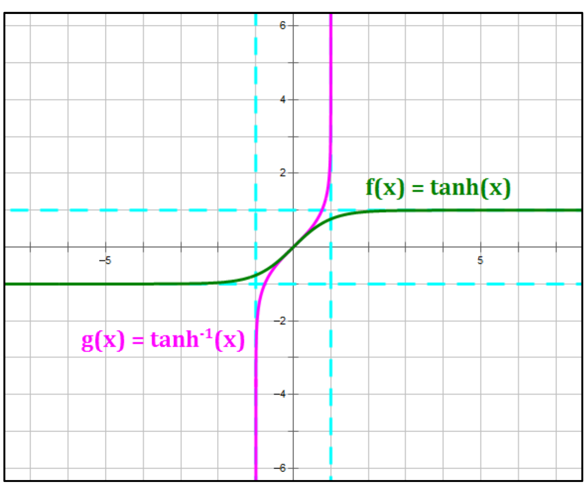
Graphs of Hyperbolic Functions and Their Inverses

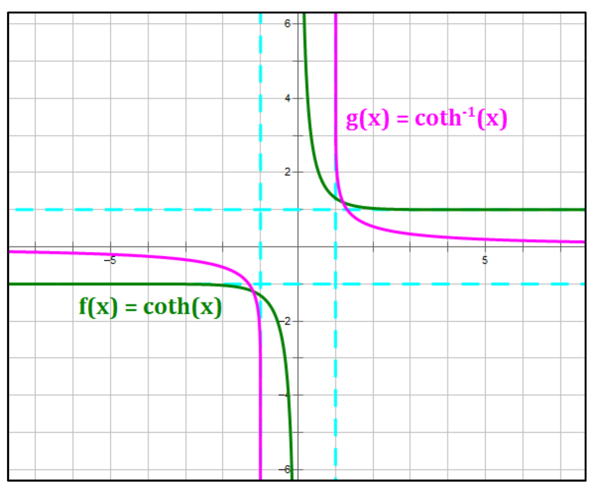












Derivatives of Hyperbolic Functions and Their Inverses

Hyperbolic Functions

$\dfrac{d}{dx}\sinh x = \cosh x$	$\dfrac{d}{dx}\sinh u = \cosh u \cdot \dfrac{du}{dx}$
$\dfrac{d}{dx}\cosh x = \sinh x$	$\dfrac{d}{dx}\cosh u = \sinh u \cdot \dfrac{du}{dx}$
$\dfrac{d}{dx}\tanh x = \operatorname{sech}^2 x$	$\dfrac{d}{dx}\tanh u = \operatorname{sech}^2 u \cdot \dfrac{du}{dx}$
$\dfrac{d}{dx}\coth x = -\operatorname{csch}^2 x$	$\dfrac{d}{dx}\coth u = -\operatorname{csch}^2 u \cdot \dfrac{du}{dx}$
$\dfrac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x\,\tanh x$	$\dfrac{d}{dx}\operatorname{sech} u = -\operatorname{sech} u\,\tanh u \cdot \dfrac{du}{dx}$
$\dfrac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x\coth x$	$\dfrac{d}{dx}\operatorname{csch} u = -\operatorname{csch} u\coth u \cdot \dfrac{du}{dx}$

Inverse Hyperbolic Functions

$\dfrac{d}{dx}\sinh^{-1} x = \dfrac{1}{\sqrt{x^2+1}}$	$\dfrac{d}{dx}\sinh^{-1} u = \dfrac{1}{\sqrt{u^2+1}} \cdot \dfrac{du}{dx}$
$\dfrac{d}{dx}\cosh^{-1} x = \dfrac{1}{\sqrt{x^2-1}}$	$\dfrac{d}{dx}\cosh^{-1} u = \dfrac{1}{\sqrt{u^2-1}} \cdot \dfrac{du}{dx}$
$\dfrac{d}{dx}\tanh^{-1} x = \dfrac{1}{1-x^2}$	$\dfrac{d}{dx}\tanh^{-1} u = \dfrac{1}{1-u^2} \cdot \dfrac{du}{dx}$
$\dfrac{d}{dx}\coth^{-1} x = \dfrac{1}{1-x^2}$	$\dfrac{d}{dx}\coth^{-1} u = \dfrac{1}{1-u^2} \cdot \dfrac{du}{dx}$
$\dfrac{d}{dx}\operatorname{sech}^{-1} x = \dfrac{-1}{x\sqrt{1-x^2}}$	$\dfrac{d}{dx}\operatorname{sech}^{-1} u = \dfrac{-1}{u\sqrt{1-u^2}} \cdot \dfrac{du}{dx}$
$\dfrac{d}{dx}\operatorname{csch}^{-1} x = \dfrac{-1}{\lvert x\rvert\sqrt{1+x^2}}$	$\dfrac{d}{dx}\operatorname{csch}^{-1} u = \dfrac{-1}{\lvert u\rvert\sqrt{1+u^2}} \cdot \dfrac{du}{dx}$

Integrals of Hyperbolic Functions and Their Inverses

Hyperbolic Functions

Be careful with these integrals. A couple of them have inverse trigonometric functions in the formulas. These are highlighted in blue.

$$\int \sinh u\, du = \cosh u$$

$$\int \cosh u\, du = \sinh u$$

$$\int \tanh u\, du = \ln(\cosh u)$$

$$\int \operatorname{sech}^2 u\, du = \tanh u$$

$$\int \coth u\, dx = \ln|\sinh u|$$

$$\int \operatorname{csch}^2 u\, du = -\coth u$$

$$\int \operatorname{sech} u\, du = 2\tan^{-1}(e^u)$$

$$\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u$$

$$\int \operatorname{csch} u\, du = \ln\left|\tanh\left(\frac{u}{2}\right)\right|$$

$$\int \operatorname{csch} u \coth u\, du = -\coth u$$

Inverse Hyperbolic Functions

$$\int \sinh^{-1} u\, du = u\sinh^{-1} u - \sqrt{u^2+1}$$

$$\int \cosh^{-1} u\, du = u\cosh^{-1} u + \sqrt{u^2-1}$$

$$\int \tanh^{-1} u\, du = u\tanh^{-1} u + \frac{1}{2}\ln(1-u^2)$$

$$\int \coth^{-1} u\, du = u\coth^{-1} u + \frac{1}{2}\ln(u^2-1)$$

$$\int \operatorname{sech}^{-1} u\, du = u\operatorname{sech}^{-1} u + \sin^{-1} u$$

$$\int \operatorname{csch}^{-1} u\, du = u\operatorname{csch}^{-1} u + \sinh^{-1} u \quad \text{if } u > 0$$

$$= u\operatorname{csch}^{-1} u - \sinh^{-1} u \quad \text{if } u < 0$$

Note: the integration rules presented in this chapter omit the " $+\,C$" term that must be added to all indefinite integrals in order to save space and avoid clutter. Please remember to add the " $+\,C$" term on all work you perform with indefinite integrals.

Other Integrals Relating to Hyperbolic Functions

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\frac{x}{a} \qquad \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\frac{x}{a} \qquad \int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left|x+\sqrt{x^2-a^2}\right|$$

$$\left.\begin{aligned} \int \frac{1}{a^2-x^2}\,dx\,\Big|_{a>x} &= \frac{1}{a}\tanh^{-1}\frac{x}{a} \\ \int \frac{1}{a^2-x^2}\,dx\,\Big|_{x>a} &= \frac{1}{a}\coth^{-1}\frac{x}{a} \end{aligned}\right\} \quad \int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|$$

$$\int \frac{1}{x\sqrt{a^2-x^2}}\,dx = -\frac{1}{a}\operatorname{sech}^{-1}\frac{|x|}{a} \qquad \int \frac{1}{x\sqrt{a^2-x^2}}\,dx = -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2-x^2}}{|x|}\right)$$

$$\int \frac{1}{x\sqrt{a^2+x^2}}\,dx = -\frac{1}{a}\operatorname{csch}^{-1}\frac{|x|}{a} \qquad \int \frac{1}{x\sqrt{a^2+x^2}}\,du = -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2+x^2}}{|x|}\right)$$

Note: The results above are shown without their constant term $(+C)$. When more than one result is shown, the results may differ by a constant, meaning that the constants in the formulas may be different.

Example 6.2: From the first row above:

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\frac{x}{a}+C_1 \quad \text{and} \quad \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)+C_2$$

From earlier in this chapter, we know that the logarithmic form of $\sinh^{-1} x$ is:

$$\sinh^{-1} x = \ln\left(x+\sqrt{x^2+1}\right)$$

Then:

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\frac{x}{a}+C_1 = \ln\left(\frac{x}{a}+\sqrt{\left(\frac{x}{a}\right)^2+1}\right)+C_1$$

$$= \ln\left(\frac{x+\sqrt{x^2+a^2}}{a}\right)+C_1 = \ln\left(x+\sqrt{x^2+a^2}\right)-\ln a+C_1$$

So we see that $C_2 = -\ln a + C_1$ and so the formulas both work, but have different constant terms.

Definite Integrals as Riemann Sums

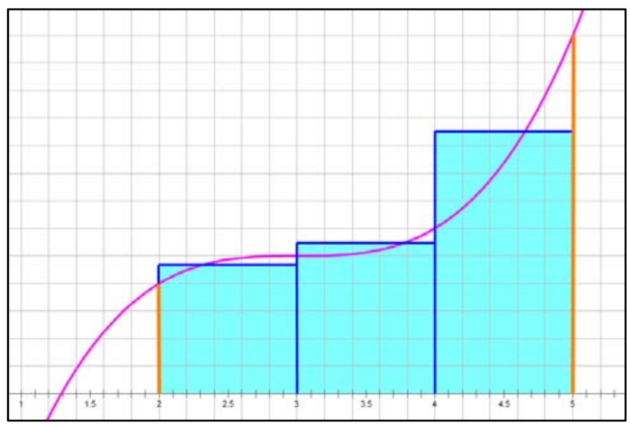
Riemann Sum

A Riemann Sum is the sum of the areas of a set of rectangles that can be used to approximate the area under the curve over a closed interval.

Note: the term "under the curve" is generally used to refer to the area between a curve and the x-axis. Some of this area may not be strictly under the curve if the curve is below the x-axis.

Consider a closed interval $[a, b]$ on x that is partitioned into n sub-intervals of lengths $\Delta x_1,\ \Delta x_2,\ \Delta x_3,\ \ldots\ \Delta x_n$. Let x_i^* be any value of x on the i-th sub-interval. Then, the Riemann Sum is given by:

$$S = \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x_i$$



A graphical representation of a Riemann sum on the interval $[2, 5]$ is provided at right.

Note that the area under the curve from $x = a$ to $x = b$ is:

$$\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x_i = \int_a^b f(x)\, dx$$

The largest Δx_i is called the mesh size of the partition. A typical Riemann Sum is developed with all Δx_i the same (i.e., constant mesh size), but this is not required. The resulting definite integral, $\int_a^b f(x)\, dx$ is called the Riemann Integral of $f(x)$ on the interval $[a, b]$.

With constant mesh size, the Riemann Integral of $f(x)$ on the interval $[a, b]$ can be expressed:

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x = \int_a^b f(x)\, dx$$

where,

$$\Delta x = \frac{\text{interval length}}{\text{number of subintervals}}.$$

Methods for Calculating Riemann Sums

Riemann Sums are often calculated using equal sub-intervals over the interval specified. Below are examples of 4 commonly used approaches. Although some methods provide better answers than others under various conditions, the limits under each method as $\max \Delta x_i \to 0$ are the same, and are equal to the integral they are intended to approximate.

Example 7.1: Given: $f(x) = \int_2^8 \left(x^2 - x\right) dx$. Using $n = 3$, estimate the area under the curve.

$\Delta \boldsymbol{x} = \frac{8-2}{3} = \mathbf{2}$. The three intervals in question are: $[\mathbf{2}, \mathbf{4}]$, $[\mathbf{4}, \mathbf{6}]$, $[\mathbf{6}, \mathbf{8}]$. Then,

$$A = \sum_{i=1}^{3} f(x_i) \cdot \Delta \boldsymbol{x_i} = \Delta \boldsymbol{x} \cdot \sum_{i=1}^{3} f(x_i)$$

Left-Endpoint Rectangles (use rectangles with left endpoints on the curve)

$$\boldsymbol{L} = \mathbf{2} \cdot [f(2) + f(4) + f(6)] = \mathbf{2} \cdot (2 + 12 + 30) = \mathbf{88} \text{ units}^2$$

Right-Endpoint Rectangles (use rectangles with right endpoints on the curve)

$$\boldsymbol{R} = \mathbf{2} \cdot [f(4) + f(6) + f(8)] = \mathbf{2} \cdot (12 + 30 + 56) = \mathbf{196} \text{ units}^2$$

Trapezoid Rule (use trapezoids with all endpoints on the curve)

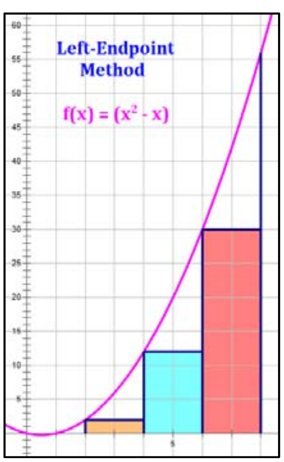
$$\boldsymbol{T} = \frac{\boldsymbol{L} + \boldsymbol{R}}{\mathbf{2}} = \frac{\mathbf{88 + 196}}{\mathbf{2}} = \mathbf{142} \text{ units}^2$$

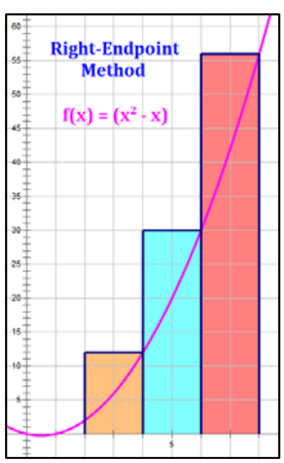
Note: the actual value of the area under the curve is:

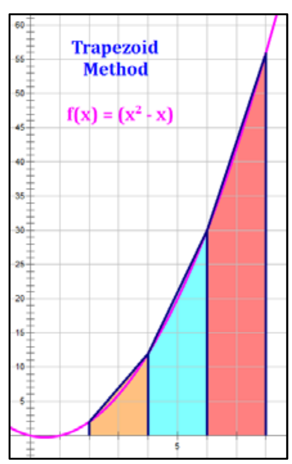
$$\int_2^8 (x^2 - x)\, dx = 138$$

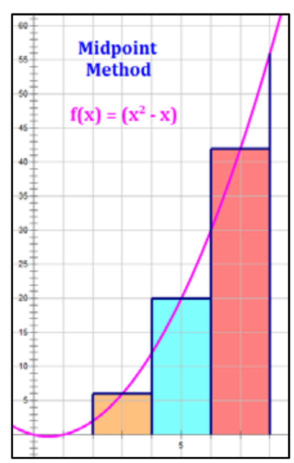
Midpoint Rule (use rectangles with midpoints on the curve)

$$\boldsymbol{M} = \mathbf{2} \cdot [f(3) + f(5) + f(7)] = \mathbf{2} \cdot (6 + 20 + 42) = \mathbf{136} \text{ units}^2$$









Riemann Sums of Tables Using the TI-84

Consider the following problem:

Use a right Riemann Sum to approximate the area under the curve on the interval $[2, 13]$.

x	2	4	5.5	8	9.2	10.3	11.8	13
$f(x)$	4	-1	-2	1	5	11	13	9

There are eight columns and, therefore, seven intervals in the table. The formula for the required Riemann Sum, then, is:

$$A = \sum_{i=1}^{7} f(x_i) \cdot \Delta x_i$$

where the Δx_i are the widths of the intervals and the $f(x_i)$ are the values of the function at the right side of each interval (because we are calculating a right Riemann Sum).

The student can calculate this directly as:

$$A = (-1)(4-2) + (-2)(5.5-4) + 1(8-.5.5) + 5(9.2-8) + 11(10.3-9.2) + 13(11.8-10.3) + 9(13-11.8) = \mathbf{45.9}$$

Alternatively, the student can use the TI-84 calculator as follows:

Step 1: **STAT – EDIT – L1** – enter the values of Δx_i in the column for L1.

Step 2: **STAT – EDIT – L2** – enter the appropriate values of $f(x_i)$ in the column for L2.

Step 3: **2ND – QUIT** – this will take you back to the TI-84's home screen.

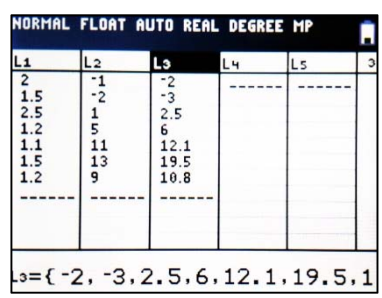
Step 3: **L1 x L2 STO> L3** – this will put the product of columns L1 and L2 in column L3. Note that L3 will contain the areas of each of the rectangles in the Riemann Sum.

Step 4: **2ND – LIST – MATH – SUM(– L3** – this will add the values in column L3, giving the value of A, which, for this problem, matches the sum of $\mathbf{45.9}$ shown above.

Note: entering **L1**, **L2** or **L3** requires use of the **2ND** key.

The student can review the contents of the lists **L1**, **L2**, and **L3** using **STAT – EDIT**. For this problem, the display will look something like the image at right. The advantages of this are:

- It allows the student to check their work quickly.
- If the student is asked for some other kind of Riemann Sum, a portion of the required input is already in the TI-84.



Each student should use whichever method of calculating Riemann Sums works best for them.

Riemann Integrals with Constant Mesh Size

With constant mesh size, the Riemann Integral of $f(x)$ on the interval $[a, b]$ can be expressed:

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\cdot\Delta x = \int_a^b f(x)dx$$

where,

$$\Delta x = \frac{\text{interval length}}{\text{number of intervals}}.$$

A formula for a right-endpoint Riemann Sum with n sub-intervals, then, can be developed using:

$$x_i^* = \text{right endpoint} = \text{left endpoint} + i\cdot\Delta x,\quad i = 1, 2, \dots, n$$

Example 7.2: Provide the exact value of $\int_2^4 x^2\,dx$ by expressing it as the limit of a Riemann sum.

Start with the definition of a Riemann Sum with constant mesh size (see above):

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\cdot\Delta x$$

Using n sub-intervals and a right-endpoint Riemann Sum,

$$\Delta x = \frac{4-2}{n} = \frac{2}{n}$$

$$x_i^* = 2 + i\left(\frac{2}{n}\right) = 2 + \frac{2i}{n},\quad i = 1, 2, \dots, n$$

$$f(x_i^*) = f\left(2 + \frac{2i}{n}\right) = \left(2 + \frac{2i}{n}\right)^2 = 4 + \frac{8i}{n} + \frac{4i^2}{n^2}$$

So,

$$\int_2^4 x^2 dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(2+\frac{2i}{n}\right)^2\right]\cdot\left(\frac{2}{n}\right) = \boldsymbol{\lim_{n\to\infty}\sum_{i=1}^{n}\left(4+\frac{8i}{n}+\frac{4i^2}{n^2}\right)\cdot\left(\frac{2}{n}\right)}$$

Riemann Sum Methods – Over- or Under-Estimates

Left and Right Endpoint Methods

Over- or Under-estimates for the Left and Right Endpoint Methods depend on whether a function is increasing or decreasing over the interval used.

Increasing

Method	Over- or Under-Estimate
Left Endpoint	Under
Right Endpoint	Over

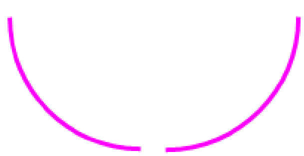
Decreasing

Method	Over- or Under-Estimate
Left Endpoint	Over
Right Endpoint	Under

Midpoint and Trapezoid Methods

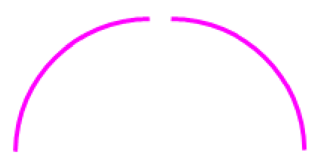
Over- or Under-estimates for the Midpoint and Trapezoidal Methods depend on whether a function is concave up or concave down over the interval used.

Concave Up



Method	Over- or Under-Estimate
Midpoint	Under
Trapezoidal	Over

Concave Down



Method	Over- or Under-Estimate
Midpoint	Over
Trapezoidal	Under

Rules of Definite Integration

First Fundamental Theorem of Calculus

If $f(x)$ is a continuous function on $[a, b]$, and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Second Fundamental Theorem of Calculus

If $f(x)$ is a continuous function on $[a, b]$, then for every $x \in [a, b]$

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Chain Rule of Definite Integration

If $f(x)$ is a continuous function on $[a, b]$, then for every $x \in [a, b]$

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f\big(g(x)\big) \cdot \frac{d}{dx}g(x)$$

Mean Value Theorem for Integrals

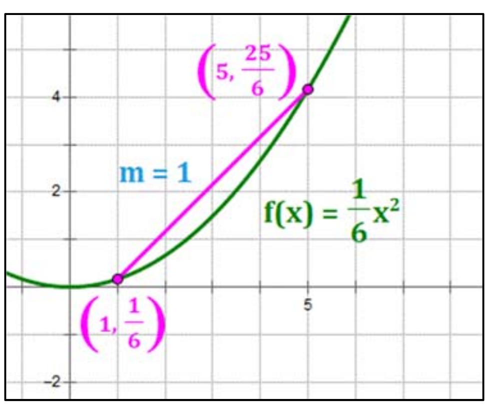
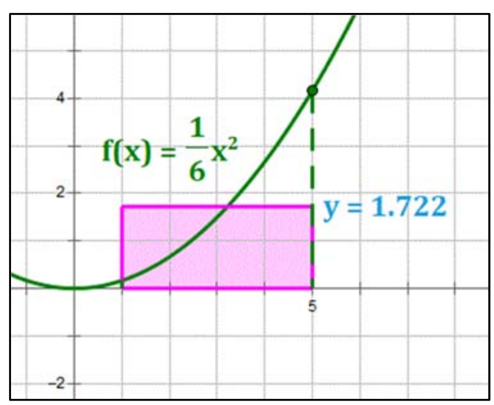
If $f(x)$ is a continuous function on $[a, b]$, then there is a value $c \in [a, b]$, such that

$$\int_a^b f(x)\,dx = (b - a) \cdot f(c)$$

The value $f(c)$ is called the Average Value of the function $f(x)$ on the interval $[a, b]$. A formula for the average value is:

$$\text{Average Value} = \frac{1}{b - a}\int_a^b f(x)\,dx$$

Average Rate of Change vs. Average Value

	Average Rate of Change	Average Value
Math Level	Algebra 1	Integral Calculus
Requirement	$f(x)$ is continuous on $[a, b]$	$f(x)$ is continuous on $[a, b]$
Description	Slope of the secant line connecting the endpoints of the curve on the interval $[a, b]$	Height of the rectangle with area equal to the area under the curve on the interval $[a, b]$
Formula	$\dfrac{f(b) - f(a)}{b - a}$	$\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$
Illustration	Average rate of change of $f(x) = \frac{1}{6}x^2$ over the interval $[1, 5]$. $\dfrac{\frac{1}{6}(5)^2 - \frac{1}{6}(1)^2}{5 - 1} = 1$  	Average value of $f(x) = \frac{1}{6}x^2$ on the interval $[1, 5]$. $\dfrac{1}{5-1}\displaystyle\int_1^5 \left(\frac{1}{6}x^2\right) dx = 1.722$ 

Properties of Definite Integrals

Same Upper and Lower Limits

$$\int_a^a f(x)\,dx = 0$$

If the upper and lower limits of the integral are the same, its value is zero.

Reversed Limits

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

Reversing the limits of an integral negates its value.

Multiplication by a Scalar

$$\int_a^b k\,f(x)\,dx = k\cdot\int_a^b f(x)\,dx$$

The integral of the product of a scalar and a function is the product of the scalar and the integral of the function.

Telescoping Limits

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

The integral over the interval $[a, b]$ is equal to the integral over the interval $[a, c]$, plus the integral over the interval $[c, b]$.

Sum or Difference

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

The integral of a sum (or difference) of functions is the sum (or difference) of the integrals of the functions.

Linear Combination

$$\int_a^b [k\cdot f(x) + m\cdot g(x)]\,dx = k\cdot\int_a^b f(x)\,dx + m\cdot\int_a^b g(x)\,dx$$

The integral of a linear combination of functions is the linear combination of the integrals of the functions.

Solving Definite Integrals with Directed Line Segments

A common problem in elementary Calculus is to use the values of definite integrals of a given function $f(x)$ over two or more intervals to obtain the value of a definite integral of $f(x)$ over a related interval. The illustration below shows how directed line segments can be used to simplify the calculations required for this kind of problem.

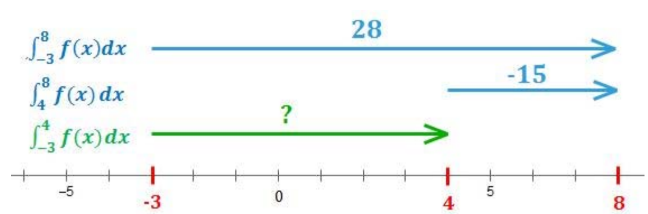
Example 7.3: Given that $\int_{-3}^{8} 3f(x)dx = 84$ and $\int_{4}^{8} 5f(x)\ dx = -75$, find $\int_{-3}^{4} f(x)\, dx$.

Step 1: Remove any scalar multipliers by dividing the values given by the scalar multipliers.

Divide: $\int_{-3}^{8} 3f(x)dx = 84$ by 3 to get $\int_{-3}^{8} f(x)\, dx = 28.$

Divide: $\int_{4}^{8} 5f(x)\ dx = -75$ by 5 to get $\int_{4}^{8} f(x)\, dx = -15.$

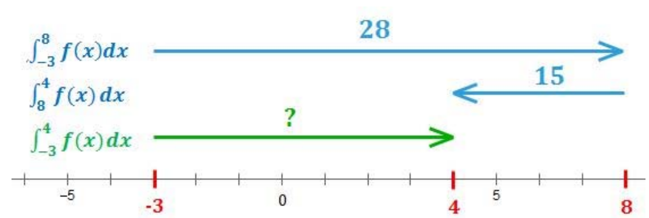
Step 2: Draw directed line segments for each of the definite integrals in the problem. Label each segment with its magnitude. The starting and ending points of each segment reflect the limits in the integral. Known values are shown in blue and the target value is in green.



Notice that the first segment stretches over the interval $[-3, 8]$ and has magnitude 28, reflecting $\int_{-3}^{8} f(x)\, dx = 28.$ The other segments are constructed similarly. We want to find the magnitude of the third (green) segment.

We could subtract the second segment from the first to obtain the solution segment. Its magnitude would be: $\int_{-3}^{4} f(x)\, dx = 28 - (-15) = \mathbf{43}.$ If we do this, we are done; we have our solution. Alternatively, we could take a more fluid approach to this problem as in Step 3.

Step 3 (if desired): Reorient segments as needed so we can follow the known directed segments from the beginning to the end of the interval required for the solution (i.e., from $x = -3$ to $x = 4$).



If we reorient the middle segment so it is pointing to the left, the magnitude of the new second segment becomes 15, reflecting the fact that we are moving to the left instead of to the right. Using Calculus, this reflects the fact that $\int_{8}^{4} f(x)\, dx = -\int_{4}^{8} f(x)\, dx = 15.$ We are now able to get to $x = 4$ by following the known segments in the directions shown. Then, we simply add the magnitudes of the known segments to get our solution: $\int_{-3}^{4} f(x)\, dx = 28 + 15 = \mathbf{43}.$

Definite Integrals – u-Substitution

u-substitution may be used in the evaluation of definite integrals as well as indefinite integrals (note: using u-substitution with indefinite integrals is covered in Chapter 5). The process with definite integrals is slightly different and may even be a bit easier.

Process

Following are the steps for the general solution to a definite integral using u–substitution.

1. Set a portion of the integrand equal to a new variable, e.g., u. Look to the rest of the integrand in deciding what to set equal to u. You will need to have du in the integrand as well, if this technique is to find success.
2. Find du in terms of dx.
3. Rearrange the integrand so that the integral exists in terms of u instead of x.
4. Perform the integration.
5. Evaluate the values of the limits of integration in terms of the new variable and substitute these into the definite integral in terms of u.
6. Evaluate the result.

Note that by calculating the limits of integration in terms of the new variable, u, we are able to avoid the step where we must substitute the expression for u back into the result of the integration. This saves time and reduces the likelihood of error in the calculation.

Example 7.4: Evaluate: $\int_{-1}^{0} \frac{2\,dx}{(2x-1)^2}$

$$\int_{-1}^{0} \frac{2\,dx}{(2x-1)^2}$$

$$\boxed{\begin{aligned} &u = 2x - 1 \\ &du = 2\,dx \\ &x = 0 \quad \Rightarrow \quad u = -1 \\ &x = -1 \quad \Rightarrow \quad u = -3 \end{aligned}}$$

$$= \int_{-1}^{0} \frac{1}{(2x-1)^2}\, 2\,dx$$

$$= \int_{-3}^{-1} u^{-2}\, du$$

$$= -u^{-1}\Big|_{-3}^{-1} = -\frac{1}{u}\Big|_{-3}^{-1} = -\left(\frac{1}{-1} - \frac{1}{-3}\right) = -\left(-1 + \frac{1}{3}\right) = \mathbf{\frac{2}{3}}$$

Example 7.5: Evaluate: $\int_{-\pi}^{\pi/4} \sin 2x \, dx$

$$\int_{-\pi}^{\pi/4} \sin 2x \; dx$$

$$= \frac{1}{2} \int_{-\pi}^{\pi/4} \sin 2x \; 2\,dx$$

$$= \frac{1}{2} \int_{-2\pi}^{\pi/2} \sin u \; du$$

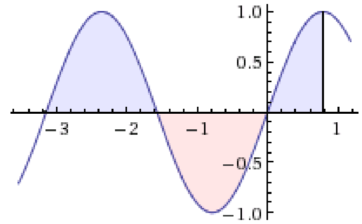
$$= \frac{1}{2} (-\cos u) \Big|_{-2\pi}^{\pi/2} = -\frac{1}{2} \cos u \Big|_{-2\pi}^{\pi/2} = -\frac{1}{2} (0 - 1) = \mathbf{\frac{1}{2}}$$

$u = 2x$

$du = 2\,dx$

$x = \frac{\pi}{4} \;\Rightarrow\; u = \frac{\pi}{2}$

$x = -\pi \;\Rightarrow\; u = -2\pi$



Example 7.6: Evaluate: $\int_{0}^{\pi/4} \tan x \sec^2 x \, dx$

For trig functions other than sine and cosine, we need to *make sure the denominators of the functions are not zero within our interval.* If they are zero, the function is not continuous on the interval and so the Fundamental Theorem of Calculus does not apply.

For the current problem, we need to make sure $\cos x \neq 0$ over the interval $\left[0, \frac{\pi}{4}\right]$ in order to use the Fundamental Theorem of Calculus. Since $\cos x = 0$ at $x = \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}$ in this neighborhood, we are okay to proceed.

$$\int_{0}^{\pi/4} \tan x \sec^2 x \, dx$$

$$= \int_0^1 u\,du = \frac{1}{2} u^2 \Big|_0^1 = \frac{1}{2} (1^2 - 0^2) = \mathbf{\frac{1}{2}}$$

$u = \tan x$

$du = \sec^2 x \; dx$

$x = \frac{\pi}{4} \;\Rightarrow\; u = 1$

$x = 0 \;\Rightarrow\; u = 0$

ALTERNATIVE APPROACH: setting $u = \sec x$

$$\int_{0}^{\pi/4} \tan x \sec^2 x \, dx$$

$$= \int_{0}^{\pi/4} \sec x \; (\sec x \; \tan x \; dx)$$

$$= \int_1^{\sqrt{2}} u\,du = \frac{1}{2} u^2 \Big|_1^{\sqrt{2}} = \frac{1}{2} \left(\sqrt{2}^2 - 1^2\right) = \mathbf{\frac{1}{2}}$$

$u = \sec x$

$du = \sec x \tan x \; dx$

$x = \frac{\pi}{4} \;\Rightarrow\; u = \sqrt{2}$

$x = 0 \;\Rightarrow\; u = 1$

Definite Integrals – Special Techniques

Sometimes it is difficult or impossible to take an antiderivative of an integrand. In such cases, it may still be possible to evaluate a definite integral, but special techniques and creativity may be required. This section presents a few techniques that the student may find helpful.

Even and Odd Functions

The following technique can sometimes be used to solve a definite integral that has limits that are additive inverses (i.e, $-a$ and a).

Every function can be split into even and odd components. The even and odd components of a given function, $f(x)$, are:

$$f_{\text{even}}(x) = \frac{f(x) + f(-x)}{2} \qquad f_{\text{odd}}(x) = \frac{f(x) - f(-x)}{2}$$

Notice that:

- $f_{\text{even}}(x) = f_{\text{even}}(-x)$, so that $f_{\text{even}}(x)$ is an even function.
- $f_{\text{odd}}(x) = -f_{\text{odd}}(-x)$, so that $f_{\text{odd}}(x)$ is an odd function.
- $f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x)$

Further recall that, for an odd function with limits that are additive inverses, any negative areas "under" the curve are exactly offset by corresponding positive areas under the curve. That is:

$$\int_{-a}^{a} f_{\text{odd}}(x)\,dx = 0$$

Additionally, for an even function with limits that are additive inverses, the area under the curve to the left of the y-axis is the same as the area under the curve to the right of the y-axis. That is:

$$\int_{-a}^{a} f_{\text{even}}(x)\,dx = \int_{-a}^{0} f_{\text{even}}(x)\,dx + \int_{0}^{a} f_{\text{even}}(x)\,dx = 2\int_{0}^{a} f_{\text{even}}(x)\,dx$$

Therefore, we have:

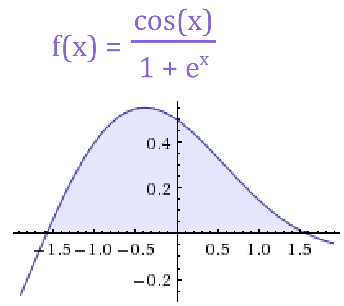
$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{a} [\, f_{\text{even}}(x) + f_{\text{odd}}(x)\,]\;dx \;=\int_{-a}^{a} f_{\text{even}}(x)\,dx \;+\; \int_{-a}^{a} f_{\text{odd}}(x)\,dx$$

And, finally, substituting from the above equations:

$$\boldsymbol{\int_{-a}^{a} f(x)\,dx \;=\; 2\int_{0}^{a} f_{\text{even}}(x)\,dx}$$

Let's look at an example of how this can be used to evaluate a difficult definite integral on the next page.

Example 7.7: **Evaluate** $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\cos x}{1+e^x}\right) dx$



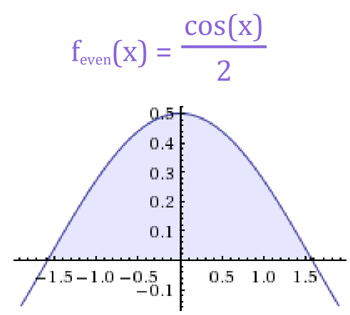
First, define: $f(x) = \dfrac{\cos x}{1+e^x}$.

Notice that there are no singularities for this integral. That is, there are no points between the limits (i.e., $-\frac{\pi}{2} < x < \frac{\pi}{2}$) at which $f(x)$ does not exist. So we may proceed in a normal fashion.

Next, let's look at the even and odd components of $f(x)$.

$$f_{\text{even}}(x) = \frac{f(x)+f(-x)}{2} = \frac{1}{2}\left[\frac{\cos x}{1+e^x} + \frac{\cos(-x)}{1+e^{-x}}\right]$$

Noting that $\cos(-x) = \cos x$, we get:



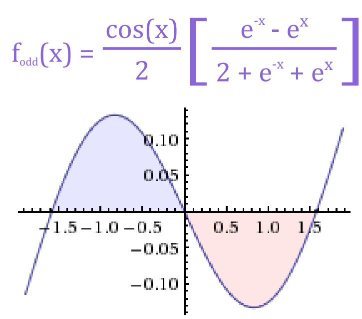
$$f_{\text{even}}(x) = \frac{1}{2}\left[\frac{\cos x}{1+e^x} + \frac{\cos x}{1+e^{-x}}\right] = \frac{\cos x}{2}\left[\frac{1}{1+e^x} + \frac{1}{1+e^{-x}}\right]$$

$$= \frac{\cos x}{2}\left[\frac{(1+e^{-x})+(1+e^x)}{(1+e^x)(1+e^{-x})}\right]$$

$$= \frac{\cos x}{2}\left[\frac{2+e^{-x}+e^x}{2+e^{-x}+e^x}\right] = \frac{\cos x}{2}$$

The odd component of $f(x)$ is (*note: this work is not necessary to evaluate the integral*):

$$f_{\text{odd}}(x) = \frac{f(x)-f(-x)}{2} = \frac{1}{2}\left[\frac{\cos x}{1+e^x} - \frac{\cos(-x)}{1+e^{-x}}\right]$$



$$= \frac{1}{2}\left[\frac{\cos x}{1+e^x} - \frac{\cos x}{1+e^{-x}}\right] = \frac{\cos x}{2}\left[\frac{1}{1+e^x} - \frac{1}{1+e^{-x}}\right]$$

$$= \frac{\cos x}{2}\left[\frac{(1+e^{-x})-(1+e^x)}{(1+e^x)(1+e^{-x})}\right]$$

$$= \frac{\cos x}{2}\left[\frac{e^{-x}-e^x}{2+e^{-x}+e^x}\right]$$

Since the value of the odd component of the definite integral is zero, we need only evaluate the even component of the definite integral using the formula on the previous page:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\cos x}{1+e^x}\right) dx = 2\int_0^{\frac{\pi}{2}} \left(\frac{\cos x}{2}\right) dx = \sin x\Big|_0^{\pi/2} = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1 - 0 = 1$$

Derivative of an Integral

The Second Fundamental Theorem of Calculus states that if $f(x)$ is a continuous function on the interval $[a,b]$, then for every $x \in [a,b]$, $\frac{d}{dx}\int_a^x f(x)dx = f(x)$. Essentially, this is a statement that integration and differentiation are inverses. But, there is more. If the upper limit is a function of x, say $u(x)$, then we must apply the chain rule to get:

$$\frac{d}{dx}\int_a^u f(t)dt \;=\; f(u)\cdot\frac{du}{dx}$$

Note that a is a constant and u is a function in x. Also note that the value of the constant a is irrelevant in this expression, as long as $f(x)$ is continuous on the required interval.

If both of the limits in the integral are functions of x, we can take advantage of a property of definite integrals to develop a solution. Let u and v both be functions in x, and let a be an arbitrary constant in the interval where $f(x)$ is continuous. Then,

$$\frac{d}{dx}\int_v^u f(t)dt \;=\; \frac{d}{dx}\int_a^u f(t)dt \;-\; \frac{d}{dx}\int_a^v f(t)dt$$

So,

$$\frac{d}{dx}\int_v^u f(t)dt \;=\; f(u)\cdot\frac{du}{dx} \;-\; f(v)\cdot\frac{dv}{dx}$$

Example 7.8:

$$\frac{d}{dx}\int_a^{3\sin 2x} t^2 dt \;=\; (3\sin 2x)^2\cdot(6\cos 2x) \;=\; \mathbf{54\ \sin^2 2x\ \cos 2x}$$

Example 7.9:

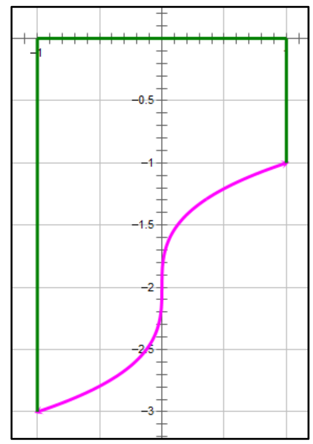
$$\frac{d}{dx}\int_{x^2}^{\tan x} e^t dt \;=\; [e^{\tan x}\cdot\sec^2 x] \;-\; \left[e^{x^2}\cdot 2x\right] \;=\; \mathbf{e^{\tan x}\sec^2 x \;-\; 2xe^{x^2}}$$

Area Under a Curve

The area under a curve can be calculated directly by integrating the curve over the desired interval. Note the following:

- The area "under" a curve is actually the area between the axis and the curve. In this sense, the word "under" may be a bit of a misnomer.
- The area under a curve may be positive (if above the x-axis) or negative (if below the x-axis).

Example 8.1: Find the area under the curve $y = t^{1/3} - 2$ on the interval $[-1, 1]$.



$$\int_{-1}^{1} \left(t^{1/3} - 2\right)\, dt = \left(\frac{3}{4} t^{4/3} - 2t\right)\Big|_{-1}^{1}$$

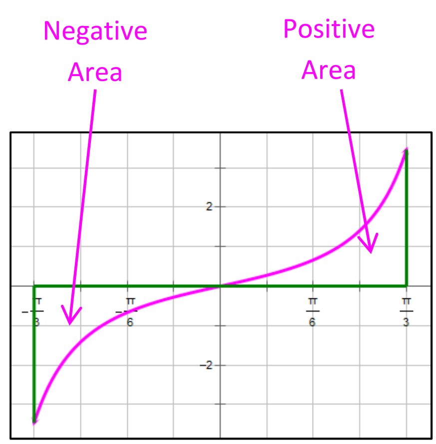
$$= \left[\frac{3}{4} \cdot \left(1^{4/3}\right) - 2 \cdot 1\right] - \left[\frac{3}{4} \cdot (-1)^{4/3} - 2 \cdot (-1)\right]$$

$$= -\frac{5}{4} - \frac{11}{4} = \mathbf{-4}$$

Example 8.2: Find the area under the curve $y = 4 \sec\theta \tan\theta$ on the interval $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$.

$$\int_{-\pi/3}^{\pi/3} (4 \sec\theta \tan\theta)\, d\theta = (4 \sec\theta)\Big|_{-\pi/3}^{\pi/3}$$

$$= \left(\frac{4}{\cos\theta}\right)\Big|_{-\pi/3}^{\pi/3} = \left(\frac{4}{\frac{1}{2}}\right) - \left(\frac{4}{\frac{1}{2}}\right) = 0$$

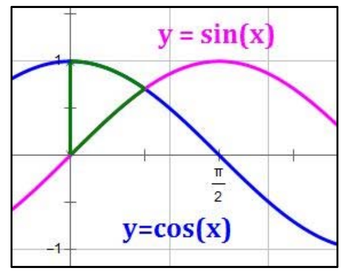


Note: this interesting result means that the negative area under the curve of $f(\theta) = 4 \sec\theta \tan\theta$ on the interval $\left[-\frac{\pi}{3}, 0\right]$ is exactly offset exactly by the positive area above the curve on the interval $\left[0, \frac{\pi}{3}\right]$.

Area Between Curves

The area between two curves is the difference of the areas under the curves. It is always positive, so if the curves switch position in terms of which one is superior (on top or to the right), the integration must take that into account.

Example 8.3: Find the area of the region is bounded by the y-axis and the curves $y = \sin x$ and $y = \cos x$ (i.e., inside the green lines in the illustration).



First, we must find the point of intersection in Quadrant 1.

$\sin x = \cos x$ at $x = \frac{\pi}{4}$, so our interval of integration is $\left[0,\ \frac{\pi}{4}\right]$

Next, consider which curve is superior to the other (i.e., which one is higher if the form of the equations is $y = f(x)$, or more to the right if the form of the equations is $x = g(y)$). The other curve is inferior. The inferior curve is subtracted from the superior curve in the integrand.

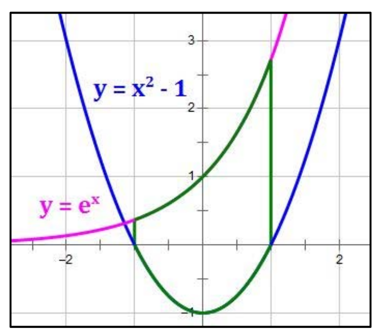
On the interval $\left[0,\ \frac{\pi}{4}\right]$, $y = \cos x$ is the higher of the two curves.

Finally, calculate the area by integrating the difference between the curves.

$$A = \int_0^{\pi/4} (\cos x - \sin x)\, dx = (\sin x + \cos x)\Big|_0^{\pi/4} = \left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - (\sin 0 + \cos 0)$$

$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0 + 1) \ = \ \mathbf{\sqrt{2} - 1}$$

Example 8.4: Find the area of the region between $y = e^x$ and $y = x^2 - 1$ on the interval $[-1,\ 1]$ (i.e., inside the green lines in the illustration).



On the interval $[-1,\ 1]$, the highest curve is $y = e^x$.

Calculate the area by integrating the difference between the curves.

$$\int_{-1}^{1} [e^x - (x^2 - 1)]\, dx = \int_{-1}^{1} (e^x - x^2 + 1)\, dx$$

$$= \left(e^x - \frac{1}{3}x^3 + x\right)\Big|_{-1}^{1}$$

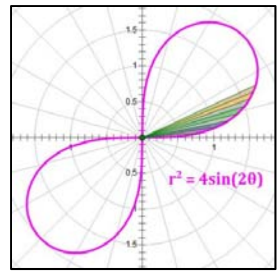
$$= \left(e - \frac{1}{3} + 1\right) - \left(\frac{1}{e} + \frac{1}{3} - 1\right) \ = \ e - \frac{1}{e} - \frac{2}{3} + 2 \ = \ \mathbf{\frac{e^2 - 1}{e} + \frac{4}{3}}$$

Area in Polar Form

Area in Polar Form is given by:

Let: $r = f(\theta)$ Then, $A = \dfrac{1}{2}\displaystyle\int_a^b r^2\, d\theta$

Why?



The diagram at right illustrates the reason that we use the above formula for area. The integral adds all of the slices (see the color slices in the diagram) inside the curve in question. Each slice is a sector of a circle with radius r and angle $d\theta$ (an infinitesimally small angle). The area of a single slice, then, is $\frac{d\theta}{2\pi}$ times the area of the circle containing it. That is:

$$A_{slice} = \frac{d\theta}{2\pi} \cdot \pi r^2 = \frac{1}{2} r^2 d\theta$$

Integrating this over the desired interval of θ results in the above formula for area.

Example 8.5: Find the area in the first quadrant inside the lemniscate $r^2 = 4\sin 2\theta$ shown in the above diagram.

First, we need to determine the limits of integration. Consider that the loop in Quadrant 1 begins and ends at locations where $r = 0$. So, we need to find two values of the variable θ that make $r = 0$. We do this by setting $r = 0$ in the equation of the lemniscate.

$0^2 = 4\sin 2\theta$, which occurs when $\sin 2\theta = 0$, which occurs at $\theta = \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots\right\}$

For our limits of integration, we will use 0 and $\frac{\pi}{2}$ because these two values define the loop in Quadrant 1. We can check this by evaluating r for a value in the interval $\left[0, \frac{\pi}{2}\right]$ and making sure the resulting point is in Quadrant 1. Let's find r when $\theta = \frac{\pi}{4}$.

$$\theta = \frac{\pi}{4} \quad \Rightarrow \quad r^2 = 4\sin\left(2 \cdot \frac{\pi}{4}\right) = 4 \cdot 1 = 4 \quad \Rightarrow \quad r = 2 \ \text{(in Quadrant 1)}$$

The area of the lemniscate above in Quadrant 1, then, is calculated as:

$$A = \frac{1}{2}\int_a^b r^2\, d\theta = \frac{1}{2}\int_0^{\pi/2} (4\sin 2\theta)\, d\theta = \int_0^{\pi/2} 2\sin 2\theta\ d\theta = -\cos 2\theta \Big|_0^{\pi/2} = \mathbf{2}$$

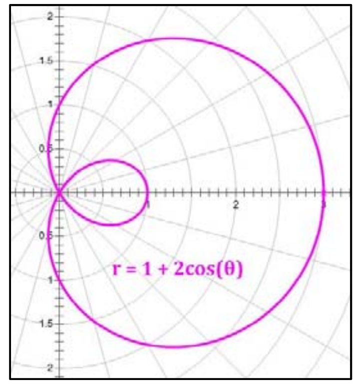
Example 8.6: Calculate the area of the general lemniscate of the form $r^2 = a^2 \sin 2\theta$.

Note that the area of the entire lemniscate is double that of the loop in Quadrant 1. Then,

$$A = 2\left(\frac{1}{2}\int_a^b r^2\, d\theta\right) = \int_0^{\pi/2} (a^2 \sin 2\theta)\, d\theta = a^2 \int_0^{\pi/2} \sin 2\theta\, d\theta = -\frac{1}{2}a^2 \cos 2\theta \Big|_0^{\pi/2} = \boldsymbol{a^2}$$

Example 8.7: Find the area within the inner loop of the limaçon $r = 1 + 2\cos\theta$.

First, we need to determine the limits of integration. Consider that the loop begins and ends at locations where $r = 0$. So, we need to find the values of the variable θ that make $r = 0$ and define the inner loop. We do this by setting $r = 0$ in the equation of the lemniscate.



$0 = 1 + 2\cos\theta$, which occurs when $\cos\theta = -\frac{1}{2}$, which occurs at $\theta = \left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$

Next, we need to make sure that the inner loop is defined as θ progresses from $\frac{2\pi}{3}$ to $\frac{4\pi}{3}$. We can do this by evaluating r for a value of θ in the interval $\left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]$ and making sure the resulting point is on the inner loop. Let's find r when $\theta = \pi$.

$$\theta = \pi \qquad \Rightarrow \qquad r = 1 + 2\cos\pi = -1$$

We check the polar point $(-1, \pi)$ on the curve and note that it is on the inner loop. Therefore, our limits of integration are the values $\theta = \left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$.

The area of the inner loop of the limaçon $r = 1 + 2\cos\theta$, then, is calculated as:

$$A = \frac{1}{2}\int_a^b r^2\, d\theta = \frac{1}{2}\int_{2\pi/3}^{4\pi/3} (1 + 2\cos\theta)^2\, d\theta = \frac{1}{2}\int_{2\pi/3}^{4\pi/3} (1 + 4\cos\theta + 4\cos^2\theta)\, d\theta$$

$$= \frac{1}{2}\int_{2\pi/3}^{4\pi/3} \left(1 + 4\cos\theta + 4\cdot\frac{1+\cos 2\theta}{2}\right) d\theta = \int_{2\pi/3}^{4\pi/3} \left(\frac{3}{2} + 2\cos\theta + \cos 2\theta\right) d\theta$$

$$= \frac{3}{2}\theta + 2\sin\theta + \frac{1}{2}\sin 2\theta \Big|_{2\pi/3}^{4\pi/3} = \boldsymbol{\pi - \frac{3\sqrt{3}}{2}}$$

Areas of Limaçons

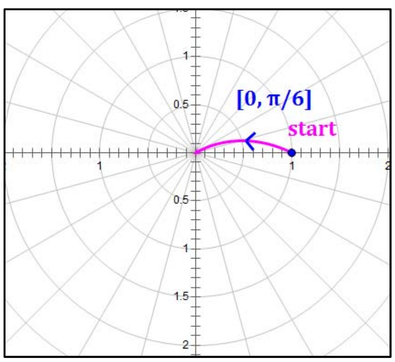
Limaçons that have both inner and outer loops present a challenge when calculating area. The general form of a limaçon is:

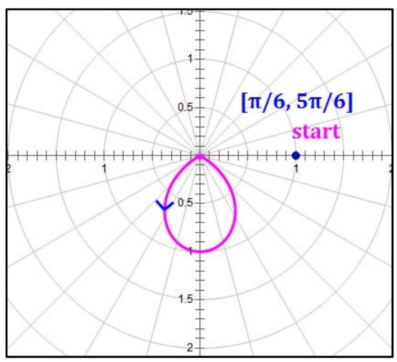
$$r = a + b\cos\theta \qquad \text{or} \qquad r = a + b\sin\theta$$

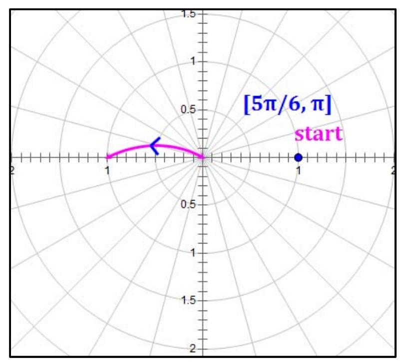
When $|a| < |b|$, the limaçon has an inner loop that covers part of its outer loop, so we must be careful calculating areas in this kind of limaçon.

Example 8.8: Find the area between the loops (i.e., inside the outer loop but outside the inner loop) of the limaçon: $r = 1 - 2\sin\theta$.

First, we need to find where $r = 1 - 2\sin\theta = 0$ so we can identify the starting and ending θ-values for the inner loop. After finding these values to be $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$, we can look at the curve over various intervals on $[0,2\pi]$ and calculate the areas associated with those intervals.





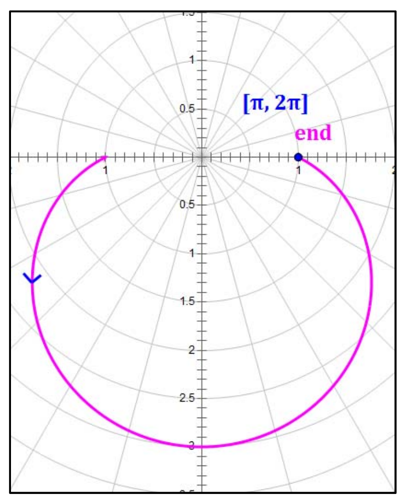


$$\left[0, \frac{\pi}{6}\right]: \quad \frac{1}{2}\int_0^{\pi/6} (1 - 2\sin\theta)^2\, d\theta = \frac{\pi + 3\sqrt{3} - 8}{4} \sim 0.0844$$

$$\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]: \quad \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1 - 2\sin\theta)^2\, d\theta = \frac{2\pi - 3\sqrt{3}}{2} \sim 0.5435$$

$$\left[\frac{5\pi}{6}, \pi\right]: \quad \frac{1}{2}\int_{5\pi/6}^{\pi} (1 - 2\sin\theta)^2\, d\theta = \frac{\pi + 3\sqrt{3} - 8}{4} \sim 0.0844$$

$$[\pi, 2\pi]: \quad \frac{1}{2}\int_{\pi}^{2\pi} (1 - 2\sin\theta)^2\, d\theta = \frac{3\pi + 8}{2} \sim 8.7124$$



The total area of the limaçon, including both the outer and inner loops, is the sum of these:

$$[0,2\pi]: \quad \frac{1}{2}\int_0^{2\pi} (1 - 2\sin\theta)^2\, d\theta = 3\pi \sim 9.4248$$

A sketch of the complete limaçon $r = 1 - 2\sin\theta$ is shown in Figure 1 below. Since taking the area from 0 to 2π includes the area completely inside the outer loop plus the area inside the inner loop, the total area can be thought of as shown in Figure 2.

This illustrates that the area within the inner loop is included in $A = \frac{1}{2}\int_0^{2\pi}(1 - 2\sin\theta)^2\, d\theta$ twice, and therefore, must be subtracted twice when looking for the area between the loops. Subtracting it once leaves all of the area inside the outer loop (Figure 3). A second subtraction is required to obtain the area between the loops.

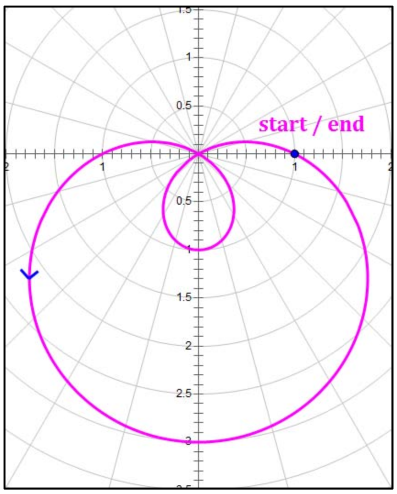


Figure 1
$r = 1 - 2\sin\theta$
Graphed on $[0, 2\pi]$

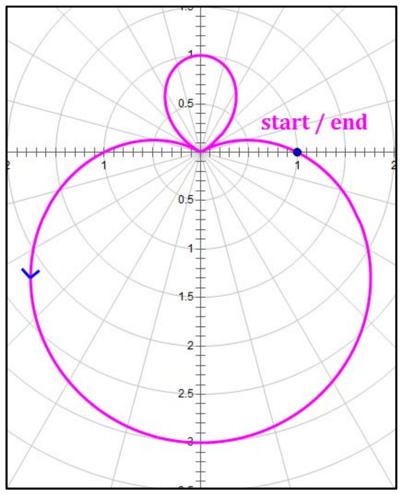


Figure 2
$r = |1 - 2\sin\theta|$
Graphed on $[0, 2\pi]$

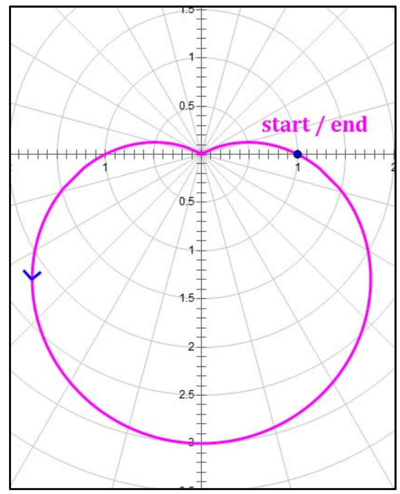


Figure 3
$r = 1 - 2\sin\theta$
Graphed on $\left[0, \frac{\pi}{6}\right] \cup \left[\frac{5\pi}{6}, 2\pi\right]$

Given all of the above, let's calculate the key areas of the limaçon $r = 1 - 2\sin\theta$:

The total area of the limaçon, including both the outer loop and the inner loop, is:

$$\text{Interval } [0{,}2\pi]: \quad \frac{1}{2}\int_0^{2\pi}(1 - 2\sin\theta)^2\, d\theta \;\; = \;\; 3\pi \;\; \sim \;\; 9.4248$$

The area inside the inner loop is calculated as:

$$\text{Interval } \left[\frac{\pi}{6}, \frac{5\pi}{6}\right]: \quad \frac{1}{2}\int_{\pi/6}^{5\pi/6}(1 - 2\sin\theta)^2\, d\theta \;\; = \;\; \frac{2\pi - 3\sqrt{3}}{2} \;\; \sim \;\; 0.5435$$

The area between the loops (i.e., the solution to this example) is calculated as:

$$\frac{1}{2}\int_0^{2\pi}(1 - 2\sin\theta)^2\, d\theta \; - \; 2 \cdot \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}(1 - 2\sin\theta)^2\, d\theta \;\; = \;\; \pi + 3\sqrt{3} \;\; \sim \;\; 8.3377$$

Arc Length

The arc length, L, of a curve, in its various forms, is discussed below:

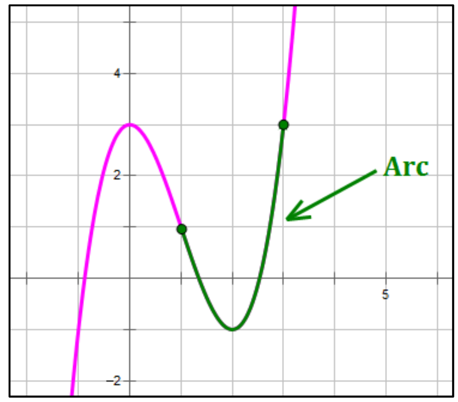
Rectangular Form:

For a function of the form: $y = f(x)$, from $x = a$ to $x = b$.

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

For a function of the form: $x = g(y)$, from $y = c$ to $y = d$.

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$



Example 8.9: Find the length of the arc on the hyperbolic curve $y = \cosh x = \frac{e^x + e^{-x}}{2}$ on the x-interval $[0, 2]$.

Using the above formula, and noting that $\frac{dy}{dx} = \frac{e^x - e^{-x}}{2} = \sinh x$:

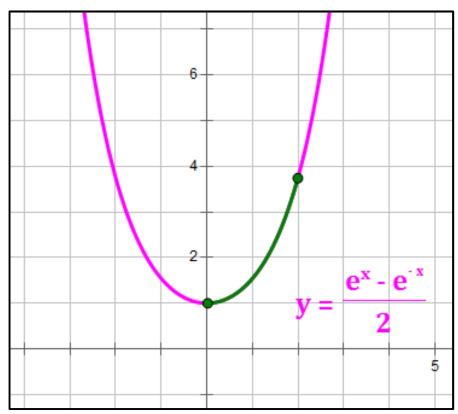
$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx = \int_0^2 \sqrt{1 + \left(\frac{e^x - e^{-x}}{2}\right)^2}\ dx$$

$$= \int_0^2 \sqrt{1 + \frac{1}{4}(e^{2x} - 2 + e^{-2x})}\ dx$$

$$= \int_0^2 \sqrt{\frac{1}{4}(e^{2x} + 2 + e^{-2x})}\ dx$$

$$= \int_0^2 \sqrt{\left(\frac{e^x + e^{-x}}{2}\right)^2}\ dx = \int_0^2 \left(\frac{e^x + e^{-x}}{2}\right) dx$$

$$= \frac{e^x - e^{-x}}{2}\Bigg|_0^2 = \frac{e^2 - e^{-2}}{2} - \frac{1 - 1}{2} = \boldsymbol{\frac{e^2 - e^{-2}}{2}} = \mathbf{\sinh 2}$$



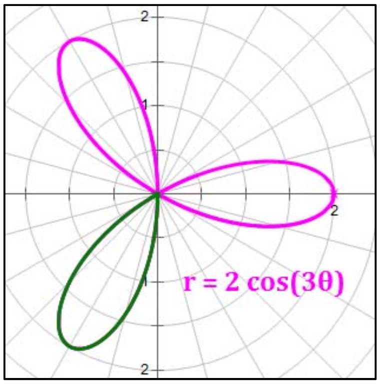
Polar Form:

For a function of the form: $r = f(\theta)$,

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example 8.10: Find the length of the arc of one petal on the rose $r = 2\cos 3\theta$.

To find the interval which defines one petal, we set $r = 0$.



$0 = 2\cos 3\theta$, which occurs when $\cos 3\theta = 0$, which occurs at $\theta = \left\{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, etc\right\}$. A little investigation reveals we can define a full petal over the interval $\theta \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$.

Next find: $\frac{dr}{d\theta} = -6\sin 3\theta$.

Then, the arc length of a single petal is:

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{\pi/6}^{\pi/2} \sqrt{(2\cos 3\theta)^2 + (-6\sin 3\theta)^2}\, d\theta$$

$$= \int_{\pi/6}^{\pi/2} \sqrt{4\cos^2 3\theta + 36\sin^2 3\theta}\, d\theta$$

$$= 2\int_{\pi/6}^{\pi/2} \sqrt{\cos^2 3\theta + 9\sin^2 3\theta}\, d\theta$$

$$= 2\int_{\pi/6}^{\pi/2} \sqrt{(\cos^2 3\theta + \sin^2 3\theta) + 8\sin^2 3\theta}\, d\theta$$

$$= 2\int_{\pi/6}^{\pi/2} \sqrt{1 + 8\sin^2 3\theta}\, d\theta$$

This expression is quite ugly but can be handled by a modern calculator. Its value is approximately **5.341** as calculated on both the TI-84 Plus and the TI nSpire.

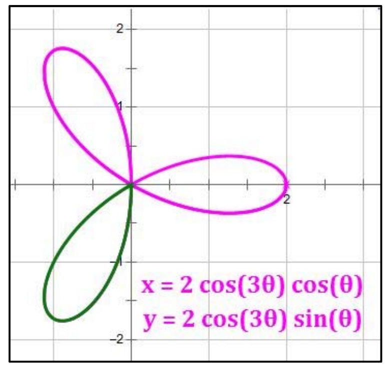
Parametric Form:

For a function of the form: $x = f(t),\ y = g(t)$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

Example 8.11: Find the length of the arc of one petal on the rose defined by the parametric equations $x = 2\cos 3\theta \cos\theta$ and $y = 2\cos 3\theta \sin\theta$.

This is the same curve defined in the example above. So we will integrate over the same interval: $\theta \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$.



To integrate in parametric form, we need $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$. Let's calculate them:

$$\frac{dx}{d\theta} = 2[(\cos 3\theta)(-\sin\theta) + (\cos\theta)(-3\sin 3\theta)]$$

$$\frac{dy}{d\theta} = 2[(\cos 3\theta)(\cos\theta) + (\sin\theta)(-3\sin 3\theta)]$$

Then,

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$= \int_{\pi/6}^{\pi/2} \sqrt{4[-(\cos 3\theta)(\sin\theta) - (\cos\theta)(3\sin 3\theta)]^2 + 4[(\cos 3\theta)(\cos\theta) - (\sin\theta)(3\sin 3\theta)]^2}\ d\theta$$

$$= 2\int_{\pi/6}^{\pi/2} \sqrt{\begin{bmatrix} (\cos^2 3\theta)(\sin^2\theta) + 6(\cos\theta)(\cos 3\theta)(\sin\theta)(\sin 3\theta) + 9(\cos^2\theta)(\sin^2 3\theta) \\ +(\cos^2 3\theta)(\cos^2\theta) - 6(\cos\theta)(\cos 3\theta)(\sin\theta)(\sin 3\theta) + 9(\sin^2\theta)(\sin^2 3\theta) \end{bmatrix}}\ d\theta$$

Notice in this expression that terms above and below each other can be combined to get:

$$L = 2\int_{\pi/6}^{\pi/2} \sqrt{[(\cos^2 3\theta)(\sin^2\theta + \cos^2\theta) + 9(\sin^2\theta + \cos^2\theta)(\sin^2 3\theta)]}\ d\theta$$

$$= 2\int_{\pi/6}^{\pi/2} \sqrt{[(\cos^2 3\theta) + 9(\sin^2 3\theta)]}\ d\theta \ = \ 2\int_{\pi/6}^{\pi/2} \sqrt{\mathbf{1 + 8\sin^2 3\theta}}\ \boldsymbol{d\theta}$$

This is exactly the same expression that was derived on the previous page in polar form.

Comparison of Formulas for Rectangular, Polar and Parametric Forms

	Rectangular Form	Polar Form	Parametric Form
Form	$y = f(x)$ $\text{position} = s(t)$	$r = f(\theta)$	$x = f(t)$ $y = g(t)$
Conversion	$x^2 + y^2 = r^2$ $\tan\theta = \dfrac{y}{x}$	$x = r\cos\theta$ $y = r\sin\theta$	$x = r\cos t$ $y = r\sin t$
Area Under Curve	$\int_a^b f(x)\,dx$	$\frac{1}{2}\int_\alpha^\beta r^2\,d\theta$	$\int_c^d [g(t)\cdot f'(t)]\,dt$
Area Between Curves	$\int_a^b [f(x) - g(x)]\,dx$	$\frac{1}{2}\int_\alpha^\beta [r_{outer}^2 - r_{inner}^2]\,d\theta$	
Arc Length (L)	$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$	$\int_\alpha^\beta \sqrt{r^2 + (r')^2}\,d\theta$	$\int_c^d \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$
Magnitude of Speed (2D)	$\lvert v(t)\rvert = \left\lvert\frac{d}{dt}[s(t)]\right\rvert$	$\sqrt{r^2 + (r')^2}$	$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$
Slope of Tangent Line	$\dfrac{dy}{dx}$	$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$	$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{r'\sin t + r\cos t}{r'\cos t - r\sin t}$
Second Derivative	$\dfrac{d^2y}{dx^2}$	$\dfrac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}}$	$\dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$
Horizontal Tangents	$\frac{dy}{dx} = 0$	$\frac{dy}{d\theta} = r'\sin\theta + r\cos\theta = 0$	$\frac{dy}{dt} = r'\sin t + r\cos t = 0$
Vertical Tangents	$\frac{dy}{dx}$ undefined	$\frac{dx}{d\theta} = r'\cos\theta - r\sin\theta = 0$	$\frac{dx}{dt} = r'\cos t - r\sin t = 0$

Area of a Surface of Revolution

Rotation about the x-Axis

Rotation of a curve $y = f(x)$ from $x = a$ to $x = b$.

$$S = \int_a^b 2\pi\, y\, ds \qquad or \qquad S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

s is the arc length of the curve on $[a, b]$.

If the curve is defined by **parametric equations,** $x = f(t)$, $y = g(t)$:

$$S = 2\pi \int_{t=t_1}^{t=t_2} y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt$$

Rotation about the y-Axis

Rotation of a curve $x = g(y)$ from $y = c$ to $y = d$.

$$S = \int_c^d 2\pi\, x\, ds \qquad or \qquad S = 2\pi \int_c^d x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

s is the arc length of the curve on $[c, d]$.

If the curve is defined by **parametric equations,** $x = f(t)$, $y = g(t)$:

$$S = 2\pi \int_{t=t_1}^{t=t_2} x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt$$

Volumes of Solids of Revolution

Solids of Revolution	Rotation about:	
	x-axis	y-axis
Disk Method	$V=\pi\int_a^b (f(x))^2\,dx$	$V=\pi\int_c^d (f(y))^2\,dy$
Washer Method(1)	$V=\pi\int_a^b \left[(f(x))^2-\left(g(x)\right)^2\right]\,dx$	$V=\pi\int_c^d [(f(y))^2-(g(y))^2]\,dy$
Cylindrical Shell Method(2)	$V=2\pi\int_c^d y\,f(y)\,dy$ or $V=2\pi\int_c^d r\,f(y)\,dy$	$V=2\pi\int_a^b x\,f(x)\,dx$ or $V=2\pi\int_a^b r\,f(x)\,dx$
Difference of Shells Method(2)(3)	$V=2\pi\int_c^d y\,(f(y)-g(y))\,dy$ or $V=2\pi\int_c^d r\,(f(y)-g(y))\,dy$	$V=2\pi\int_a^b x\left(f(x)-g(x)\right)dx$ or $V=2\pi\int_a^b r\left(f(x)-g(x)\right)dx$
Area Cross Section Method(4)	$V=\int_a^b A(x)\,dx$	$V=\int_c^d A(y)\,dy$

Notes:

1. The *Washer Method* is an extension of the *Disk Method.*
2. r is the radius of the cylindrical shell. In cases where there is a gap between the axis of revolution and the functions being revolved, r is the distance between the axis of revolution and either x or y, as appropriate.
3. The *Difference of Shells Method* is an extension of the *Cylindrical Shell Method.*
4. The function A is the area of the cross section being integrated.

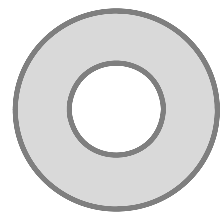
Disk and Washer Methods

The formulas for the Disk Method and Washer Method for calculating volumes of revolution are provided above. Below, we present an approach that can be used to calculate volumes of revolution using these methods.

Under the Disk Method, we integrate the area of the region between a curve and its axis of revolution to obtain volume. Since each cross-section of the resulting object will be a circle, we use the formula $area = \pi r^2$ as our starting point. The resulting formula is:

$$V = \pi \int_a^b (cross\ section\ radius)^2\, dx \quad \text{or} \quad V = \pi \int_c^d (cross\ section\ radius)^2\, dy$$

The Washer Method is simply a dual application of the Disk Method. Consider the illustration at right. If we want the area of the shaded region, we subtract the area of the smaller circle from the area of the larger circle. The same occurs with the Washer Method; since we integrate cross-sectional area to find volume, so to obtain the volume of revolution of a region between the two curves we integrate the difference in the areas between the two curves.

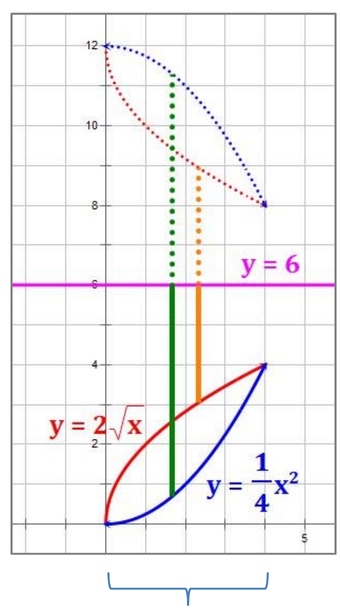


Below is a set of steps that can be used to determine the volume of revolution of a region between two curves. The approach is illustrated based on the following example:

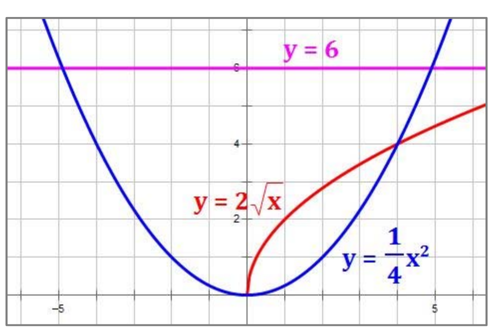
Example 8.12: Find the volume that results from revolving the region between the curves $y = 2\sqrt{x}$ and $y = \frac{1}{4}x^2$ about the line $y = 6$.

Steps

1. **Graph the equations** provided and any other information given in the problem (illustrated below). Then, isolate the section of the graph that we want to work with (illustrated at right). The disks we will use are shown as green and orange vertical lines. *The dashed objects are reflections of the curves and disks over the axis of revolution; these give us an idea of what the central cross-section of the 3D shape will look like after revolution. You do not need to draw these.*



Integration Interval



2. **Identify whether there is a gap** between the region to be revolved and the axis of revolution. In the example, the axis of revolution is $y = 6$, so there is clearly a gap between a) the red and blue curves, and b) the axis of revolution. Therefore, we will use the Washer Method.

3. **Set up the integral form** to be used.

   a. Disk Method: $V = \pi \int_a^b (\text{radius})^2\, dx$ or $V = \pi \int_c^d (\text{radius})^2\, dy$

   b. Washer Method: $V = \pi \int_a^b \left[(\text{big radius})^2 - (\text{small radius})^2\right] dx$ or $V = \pi \int_c^d \left[(\text{big radius})^2 - (\text{small radius})^2\right] dy$

4. **Identify the variable of integration** (i.e., are we using dx or dy?). The disks used must be perpendicular to the axis of revolution.

   a. If we are revolving around an axis, use the variable of that axis.

   b. If the axis of revolution is a line of the form, $x = a$ or $y = b$, use the opposite variable from the one that occurs in the equation of the axis. In the example, the axis of revolution is $y = 6$, so we will integrate with respect to x.

   Note: The expressions used in the integration must be in terms of the variable of integration. So, for example, if the variable of integration is y and the equation of a curve is given as $y = f(x)$, we must invert this to the form $x = g(y)$ before integrating.

5. **Identify the limits of integration.** In the example, the curves intersect at $x = 0$ and $x = 4$. This results in an equation for volume in the form:

$$V = \pi \int_0^4 \left[(\text{big radius})^2 - (\text{small radius})^2\right] dx$$

6. **Substitute the expressions for the big and small radii** inside the integral. In the example, we have the following:

   a. $\text{big radius} = 6 - \frac{1}{4}x^2$

   b. $\text{small radius} = 6 - 2\sqrt{x}$

   This results in the following:

$$\boldsymbol{V = \pi \int_0^4 \left[\left(6 - \frac{1}{4}x^2\right)^2 - \left(6 - 2\sqrt{x}\right)^2\right] dx \sim 140.743}$$

   Note that this matches the value calculated using the Difference of Shells Method below.

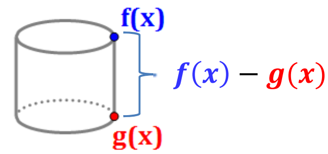
Cylindrical Shell Methods

The formulas for the Cylindrical Shell Method and Difference of Shells Method for calculating volumes of revolution are provided above. Below, we present an approach that can be used to calculate volumes of revolution using these methods.

Under the Cylindrical Shell Method, we integrate the volume of a shell across the appropriate values of x or y. We use the formula for the volume of a cylinder as our starting point (i.e., $Volume = 2\pi rh$, where h is typically the function provided). The resulting formula is:

$$V = 2\pi \int_a^b r\,(height\ of\ shell)\,dx \quad \text{or} \quad V = 2\pi \int_c^d r\,(height\ of\ shell)\,dy$$

The Difference of Shells Method is essentially a dual application of the Cylindrical Shell Method. We want the volume of the cylinder whose height is the difference between two functions (see illustration at right).

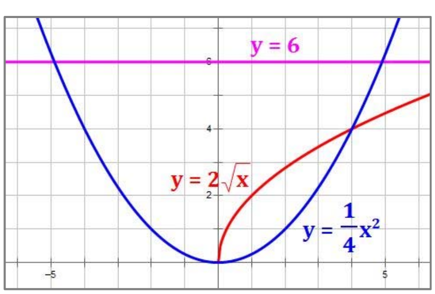


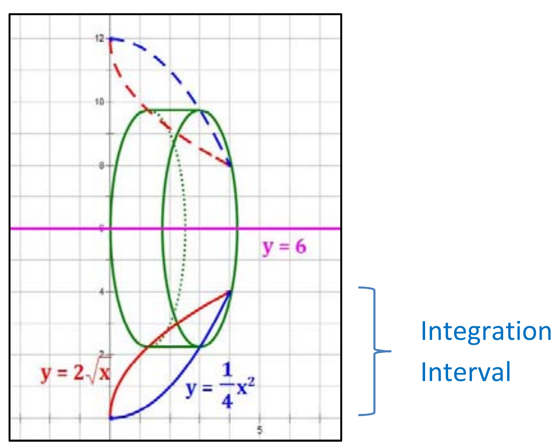
Below is a set of steps that can be used to determine the volume of revolution of a region between two curves. The approach is illustrated based on the following example:

Example 8.13: Find the volume that results from revolving the region between the curves $y = 2\sqrt{x}$ and $y = \frac{1}{4}x^2$ about the line $y = 6$.

Steps

1. **Graph the equations** provided and any other information given in the problem (illustrated below left). Then, isolate the section of the graph that we want to work with (illustrated below right). A*lso shown are reflections of the curves over the axis of revolution (dashed curves); this allows us to see the "other side" of the cylindrical shells we will use.* A typical shell is shown as a green cylinder.





2. **Identify whether the integration involves one or two curves.**
    a. One curve: Use the Cylindrical Shell Method.
    b. Two curves: Use the Difference of Shells Method. This is the case in the example.

3. **Set up the integral form** to be used. Let r be the radius of the shell.
    a. Cylindrical Shell Method: $V = 2\pi \int_a^b r\, f(x)\, dx$ or $V = 2\pi \int_c^d r\, g(y)\, dy$.
    b. Difference of Shells Method: $V = 2\pi \int_a^b r\, (\text{difference of shell heights})\, dx$ or $V = 2\pi \int_c^d r\, (\text{difference of shell heights})\, dy$.

4. **Identify the variable of integration** (i.e., are we using dx or dy?). The shells used must be <u>parallel</u> to the axis of revolution.
    a. If we are revolving around an axis, consider the equation of that axis (i.e., the x-axis has equation: $y = 0$).
    b. The axis of revolution is a line of the form, $x = a$ or $y = b$, use the same variable as the one that occurs in the equation of the axis of revolution. In the example, the axis of revolution is $y = 6$, so we will integrate with respect to y.

    $$V = 2\pi \int_c^d r\, (\text{difference of shell heights})\, dy$$

5. **Identify the limits of integration.** In the example, the curves intersect at $y = 0$ and $y = 4$. This results in an equation for volume in the form:

$$V = 2\pi \int_0^4 r\, (\text{difference of shell heights})\, dy$$

6. **Substitute the expressions for r and the difference of shell heights** into the integral. In the example, we need to convert each equation to the form $x = g(y)$ because y is the variable of integration:
    a. $y = \frac{1}{4}x^2$ so $x = 2\sqrt{y}$ $\qquad$ $y = 2\sqrt{x}$ so $x = \frac{1}{4}y^2$

    The difference of shell heights, then, is $\left(2\sqrt{y} - \frac{1}{4}y^2\right)$.
    b. The radius of a shell is the difference between the line $y = 6$ and the value of y in the interval, so the radius is $6 - y$.

This results in the following:

$$\boldsymbol{V = 2\pi \int_0^4 (6 - y)\left(2\sqrt{y} - \frac{1}{4}y^2\right) dy \sim 140.743}$$

Note that this matches the value calculated using the Washer Method above.

Volume by Area of a Cross-Section

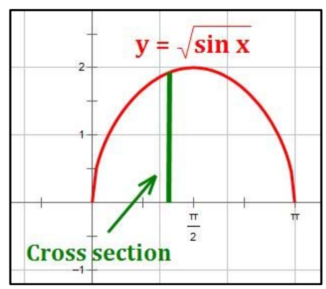
Some problems require us to determine volume of a solid using its base and cross-sectional area across that base. These are not problems based on revolution of a shape, so we use a more basic formula (that does not involve π):

$$V = \int_a^b (area\ cross\ section)\ dx \quad \text{or} \quad V = \int_c^d (area\ cross\ section)\ dy$$

Below is a set of steps that can be used to determine volume for this type of problem. The approach is illustrated using the following example:

Example 8.14: Find the volume of a solid with a base of $y = 2\sqrt{\sin x}$ over the interval $[0, \pi]$ if the cross-sections perpendicular to the x-axis are equilateral triangles whose bases stretch from the x-axis to the curve.

Steps

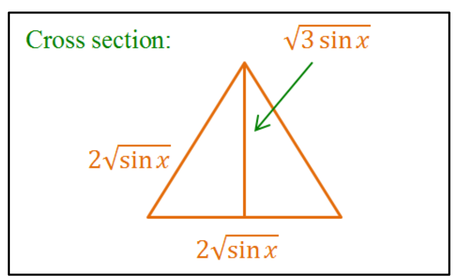


1. **Graph the curve of the base** over the interval specified.

2. **Determine the variable of integration.** This will always be the variable whose axis is perpendicular to the cross-sections specified. In the example, the variable of integration is x.

3. **Determine the limits of integration.** This is typically the interval provided in the problem. In the example, this is the interval $[0, \pi]$.

4. **Draw the cross-section** you are provided in the problem. In the example, we are working with equilateral triangles with base equal to the function $y = 2\sqrt{\sin x}$.



5. **Determine the area of the cross-section** in terms of the appropriate variable. We need the area of an equilateral triangle for this example. This area can be developed from basic principles using the illustration at right, or from the formula: $A = \frac{\sqrt{3}}{4}b^2$, where b is the length of the base of the triangle.

   In the example: $A = \frac{\sqrt{3}}{4}b^2 = \frac{\sqrt{3}}{4}\left(2\sqrt{\sin x}\right)^2 = \sqrt{3}\ \sin x$

6. **Integrate the area of the cross-section** using the limits determined in Step 3.

$$V = \int_0^\pi \sqrt{3}\ \sin x\ dx = -\sqrt{3}\ \cos x\Big|_0^\pi = \mathbf{2\sqrt{3}} \sim \mathbf{3.464}$$

Improper Integration

Improper integration refers to integration where the interval of integration contains one or more points where the integrand is not defined.

Infinite Limits

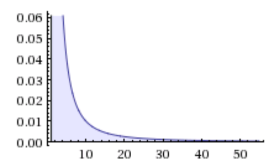
When either or both of the limits of integration are infinite, we replace the infinite limit by a variable and take the limit of the integral as the variable approaches infinity.

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx \qquad \int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx + \lim_{c\to\infty}\int_b^c f(x)\,dx$$

Note: in this third formula, you can select the value of b to be any convenient value that produces convergent intervals.

Example 9.1:

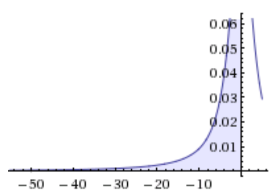


$$\int_1^\infty \left(\frac{1}{x^2}\right)dx = \lim_{a\to\infty}\int_1^a \left(\frac{1}{x^2}\right)dx$$

$$= \lim_{a\to\infty}\int_1^a (x^{-2})\,dx = \lim_{a\to\infty}(-x^{-1})\Big|_1^a$$

$$= \lim_{a\to\infty}\left(-\frac{1}{x}\right)\Big|_1^a = \lim_{a\to\infty}\left(-\frac{1}{a}+\frac{1}{1}\right) = 0+1 = 1$$

Example 9.2:



$$\int_{-\infty}^0 \left(\frac{1}{x^2+9}\right)dx = \lim_{a\to-\infty}\int_a^0 \left(\frac{1}{x^2+9}\right)dx$$

$$= \frac{1}{3}\lim_{a\to-\infty}\int_a^0 \left(\frac{1}{\left(\frac{x}{3}\right)^2+1}\right)\frac{1}{3}\,dx$$

$$= \frac{1}{3}\lim_{a\to-\infty}\left(\tan^{-1}\frac{x}{3}\right)\Big|_a^0$$

$$= \frac{1}{3}\lim_{a\to-\infty}\left(\tan^{-1}0 - \tan^{-1}\frac{a}{3}\right) = \frac{1}{3}\left[0-\left(-\frac{\pi}{2}\right)\right] = \frac{\pi}{6}$$

Discontinuous Integrand

Limits are also required in cases where the function in an integrand is discontinuous over the interval of its limits.

If there is a discontinuity at $x = a$,

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

If there is a discontinuity at $x = b$,

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

If there is a discontinuity at $x = c$ where $a < c < b$,

$$\int_a^b f(x)\,dx = \lim_{t \to c^-} \int_a^t f(x)\,dx + \lim_{t \to c^+} \int_t^b f(x)\,dx$$

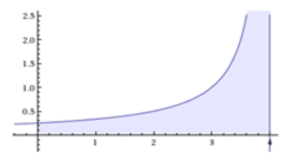
Example 9.3:

$$\int_0^4 \left(\frac{1}{4-x}\right) dx = \lim_{t \to 4^-} \int_0^t \left(\frac{1}{4-x}\right) dx$$

$$= -\lim_{t \to 4^-} [\ln(4-x)] \Big|_0^t = \lim_{t \to 4^-} [\ln(4-x)] \Big|_t^0$$

$$= \lim_{t \to 4^-} [\ln(4-0) - \ln(4-t)]$$

$$= \ln 4 \; - \; (-\infty) \; = \; +\infty$$

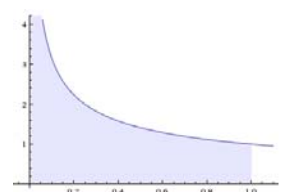


Example 9.4:

$$\int_0^1 \left(\frac{1}{\sqrt{x}}\right) dx = \int_0^1 \left(x^{-1/2}\right) dx = \lim_{t \to 0^+} \int_t^1 \left(x^{-1/2}\right) dx$$

$$= \lim_{t \to 0^+} \left(2x^{1/2}\right) \Big|_t^1 = \lim_{t \to 0^+} \left(2\sqrt{x}\right) \Big|_t^1$$

$$= \lim_{t \to 0^+} \left(2\sqrt{1} - 2\sqrt{t}\right) \; = \; 2 - 0 \; = \; 2$$

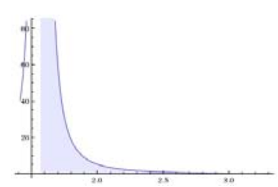


Example 9.5:

$$\int_{\pi/2}^{\pi} (\sec x \tan x)\,dx = \lim_{t \to \pi/2^+} \int_t^{\pi} (\sec x \tan x)\,dx$$

$$= \lim_{t \to \frac{\pi}{2}^+} \sec x \Big|_t^{\pi} = \lim_{t \to \frac{\pi}{2}^+} (\sec \pi - \sec t)$$

$$= \; -1 + \infty \; = \; +\infty$$



Differential Equations

Definitions

A **Differential Equation** is an equation that contains an independent variable, one or more dependent variables, and full or partial derivatives of the dependent variables.

An **Ordinary Differential Equation (ODE)** is a differential equation that contains ordinary (not partial) derivatives. Generally, an ODE is expressed in one of the following forms:

$$F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0 \qquad \text{or} \qquad F\left(x, y, y', y'', \ldots, y^{(n-1)}\right) = y^{(n)}$$

A **Partial Differential Equation (PDE)** is a differential equation that contains partial derivatives.

The **Order** of a differential equation is the highest derivative of a dependent variable in the equation.

A **Linear ODE of Order n** is an equation of the form:

$$a_n(x) \cdot y^{(n)} + a_{n-1}(x) \cdot y^{(n-1)} + \ \ldots + a_1(x) \cdot y' + a_0(x) \cdot y = f(x)$$

where each of the $a_i(x)$ is a function in x only, (i.e., not in y or any of its derivatives). The $a_i(x)$ need not be linear functions. The label "Linear" refers to y and its derivatives; that is, there are no powers of y and its derivatives and no products of y and/or any of its derivatives. For example, there are no terms like $\left(y^{(n)}\right)^2$, $(y \cdot y'')$, etc.

A **Separable** first order ODE is one that can be written in the form:

$$\frac{dy}{dx} = f(x) \cdot g(y)$$

A **Solution** to a differential equation is any function that satisfies the differential equation in the interval specified.

Initial Conditions are those that allow us to determine which of a possible set of solutions to a differential equation we seek. In essence, these allow us to determine the value of any constants that turn up in the integrations required to solve the differential equations.

An **Initial Value Problem** is a differential equation whose solution depends on the initial conditions provided.

The **Actual Solution** to a differential equation is the specific solution that satisfies both the differential equation and the initial conditions.

An **Explicit Solution** is a solution that can be expressed in the form $y = f(x)$.

An **Implicit Solution** is a solution that cannot be expressed in the form $y = f(x)$.

Separable First Order ODEs

Most of the differentiable equations that will be encountered in first year Calculus will be separable first order differential equations. Typically, we will use Algebra to identify $f(x)$ and $g(x)$ to get the equation into the form $\frac{dy}{dx} = f(x) \cdot g(y)$.

Next, we treat dy and dx as separate entities, and convert the equation to the form:

$$\frac{dy}{g(y)} = f(x)\, dx$$

Finally, we integrate both sides to obtain a solution:

$$\int \frac{dy}{g(y)} = \int f(x)\, dx$$

The final result will have a $+C$ term. Typically, you need only one $+C$ term since the constants from each integral can be subtracted to get a single constant term. Often, there is an initial condition provided which allows us to calculate the value of C.

Example 10.1: Find the explicit actual solution to $\frac{dy}{dx} = e^y$ if $(1, 0)$ is a point on the curve.

An explicit solution is one of the form "$y = f(x)$". An actual solution is one in which we have solved for any constants that pop up.

Let's begin by separating the variables.

$$\frac{dy}{dx} = e^y$$

$$e^{-y} dy = dx$$

$$\int e^{-y} dy = \int dx$$

$$-e^{-y} = x + C$$

Substituting $(1, 0)$ for (x, y) gives $-1 = 1 + C$ so, $C = -2$

$$-e^{-y} = x - 2$$

$$e^{-y} = 2 - x$$

$$-y = \ln(2 - x)$$

$\boldsymbol{y = -\ln(2 - x)}$ Note the resulting domain restriction: $x < 2$.

Example 10.2: Find the explicit actual solution to $\frac{dy}{dx} = \frac{x}{\sqrt{9+x^2}}$ if $(4,5)$ is a point on the curve.

An explicit solution is one of the form "$y = f(x)$". An actual solution is one in which we have solved for any constants that pop up.

Let's begin by separating the variables. Note that since there is an x in the numerator, we do not need to use inverse trig functions.

$$\frac{dy}{dx} = \frac{x}{\sqrt{9+x^2}}$$

$$dy = \frac{x}{\sqrt{9+x^2}}\, dx$$

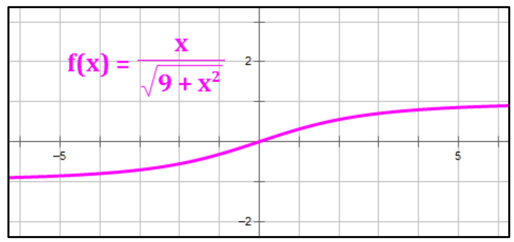
$$\int dy = \frac{1}{2}\int \frac{2x}{\sqrt{9+x^2}}\, dx$$

$$y = \frac{1}{2}\int u^{-1/2}\, du = \frac{1}{2} \cdot 2u^{1/2} + C$$

$u = 9 + x^2$

$du = 2x\, dx$

$x = 4 \;\Rightarrow\; u = 25$ and $y = 5$



$$y = \sqrt{u} + C$$

Then, substituting $(25,5)$ for (u,y) gives: $5 = \sqrt{25} + C$ so, $C = 0$

$$y = \sqrt{u} \quad \Rightarrow \quad \boldsymbol{y = \sqrt{9 + x^2}}$$

An alternative way to develop a solution, involving x more directly, would be to replace the three lines immediately above with these:

$$y = \sqrt{u} + C \quad \Rightarrow \quad y = \sqrt{9 + x^2} + C$$

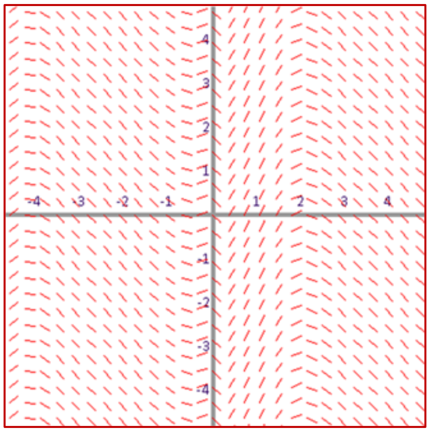
Then, substituting $(4,5)$ for (x,y) gives: $5 = \sqrt{9 + 4^2} + C \;\Rightarrow\; 5 = \sqrt{25} + C$ so, $C = 0$

$$\boldsymbol{y = \sqrt{9 + x^2}}$$

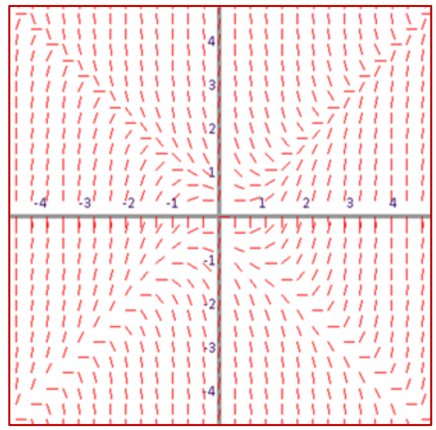
Slope Fields

A **Slope Field** (also called a **Direction Field**) is a graphical representation of the slopes of a curve at various points that are defined by a differential equation. Each position in the graph (i.e., each point (x, y)) is represented by a line segment indicating the slope of the curve at that point.

Example 10.3: $\frac{dy}{dx} = e^{\sin x} \cos x + \sin x$



Example 10.4: $\frac{dy}{dx} = x^2 - y^2$



If you know a point on a curve and if you have its corresponding slope field diagram, you can plot your point and then follow the slope lines to determine the curve.

Example 10.5: Find the explicit actual solution to $f'(x) = \frac{x}{y}$ if $(1, -2)$ is a point on the curve.

$$\frac{dy}{dx} = \frac{x}{y}$$

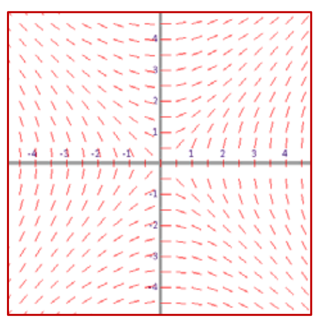
$$y\,dy = x\,dx$$

$$\int y\,dy = \int x\,dx$$

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C_1$$

$$y^2 = x^2 + C$$

Slope Field for: $\frac{dy}{dx} = \frac{x}{y}$



Slope Field generator available at:
http://www.mathscoop.com/calculus/differential-equations/slope-field-generator.php

Substituting $(1, -2)$ for (x, y) gives: $C = 3$

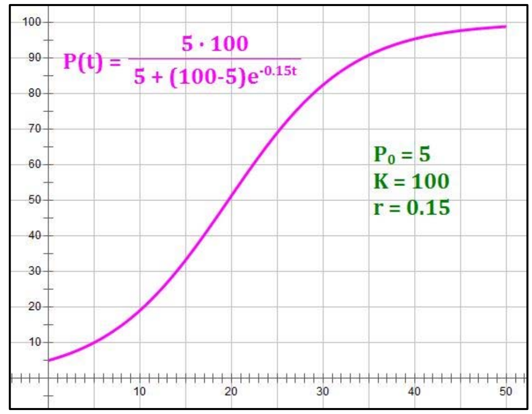
$$y^2 = x^2 + 3$$

$$y = \pm\sqrt{x^2 + 3}$$

Finally, noting that $(1, -2)$ is a solution, we can narrow the solution down to:

$$\boldsymbol{y = -\sqrt{x^2 + 3}}$$

Logistic Function



A Logistic Function describes the growth of a population over time. Early in its growth phase, the model describes near-exponential population growth. As the population grows larger, it eventually faces limits that reduce its growth rate. Late in its growth phase, a population approaches a maximum value, called the carrying capacity.

Several forms of the Logistic Function for a population $P(t)$, over time, are common:

$$P(t) = \frac{K}{1 + \left(\frac{K - P_0}{P_0}\right) e^{-rt}} \quad \text{or} \quad P(t) = \frac{P_0\, K}{P_0 + (K - P_0)e^{-rt}} \quad \text{or} \quad P(t) = \frac{K\, P_0\, e^{rt}}{K + P_0(e^{rt} - 1)}$$

The symbols in these equations have the following meanings:

- $P(t)$ is the population at time t.
- K is the carrying capacity of the population. It is the maximum population sustainable in the system
- $P_0 = P(0)$ is the initial population.
- r is the rate of growth of the population, and is called the growth parameter.
- t is the variable for time.

The differential equation that leads to the Logistic Function is:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$$

Characteristics of the Logistic Function

- $\frac{dP}{dt} > 0$ for all t
- $\lim_{t\to\infty} P(t) = K$
- $P(t)$ has an inflection point at $t = \frac{1}{r}\ln\left(\frac{K}{P_0} - 1\right)$, when $P(t) = \frac{K}{2}$. Therefore, the maximum rate of growth for the population occurs when $P(t) = \frac{K}{2}$.

Numerical Methods

If we know a point on a curve and the slope of the curve at each point, but do not know the equation of the curve, it is possible to estimate the value of another point on the same curve using numerical methods. Several of these numerical methods are presented below.

Euler's Method

Euler's Method estimates the location of the new point based on the position of the first point and the slope of the curve at intervals between the two points. Any number of intervals, n, can be used. Each interval is called a time step. The formulas involved are as follows.

Let: (x_0, y_0) be the initial (known) point.

(x_k, y_k) be the intermediate points, for $k = 1, 2, \ldots$.

(x_n, y_n) be the desired point. Note that n is the number of time steps and x_n is known.

h be the distance between successive x-values. That is, $h = \frac{x_n - x_0}{n}$.

Then, Euler's Method estimates each y_{k+1} based on y_k and the slope of the function at (x_k, y_k), using the formulas:

$$x_{k+1} = x_k + h \qquad\qquad y_{k+1} = y_k + y'(x_k) \cdot h$$

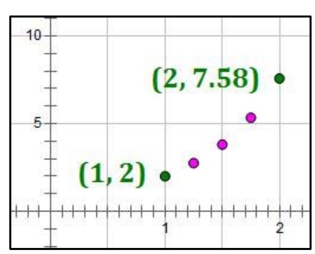
Example 10.6: Let $\frac{dy}{dx} = y'(x) = 2y - x$. Estimate $y(2)$ using 4 time steps if we know $(1, 2)$ is a point on the curve.

We start at point $(x_0, y_0) = (1,2)$, using a time step of $h = \frac{2-1}{4} = 0.25$. The following table shows the iterations required to estimate $y(2)$. Values in the table are rounded to 2 decimals for display, but the exact values are used in all calculations.

k	x_k	y_k	$y'(x_k) = 2y - x$	y_{k+1}
0	1.00	2.00	$2\,(2.00) - 1.00 = 3.00$	$2.00 + 3.00\,(0.25) = 2.75$
1	1.25	2.75	$2\,(2.75) - 1.25 = 4.25$	$2.75 + 4.25\,(0.25) = 3.81$
2	1.50	3.81	$2\,(3.81) - 1.50 = 6.13$	$3.81 + 6.13\,(0.25) = 5.34$
3	1.75	5.34	$2\,(5.34) - 1.75 = 8.94$	$5.34 + 8.94\,(0.25) = 7.58$
4	2.00	7.58		

Since it is natural to develop Euler's Method in table form, it is relatively easy to adapt it to a spreadsheet program such as Microsoft Excel.

A plot of successive values of x_k is shown in the graph at right.



Modified Euler's Method

The Modified Euler's Method is like Euler's Method, but develops the slope at each point as the average of the slopes at the beginning and end of each interval. Using the same notation as on the previous page, the Modified Euler's Method uses a two-step formula:

Predictor step: $x_{k+1} = x_k + h$ $\qquad y_{k+1} = y_k + y'(x_k) \cdot h$

Corrector step: $y_{k+1} = y_k + \frac{1}{2}[y'(x_k) + y'(x_{k+1})] \cdot h$

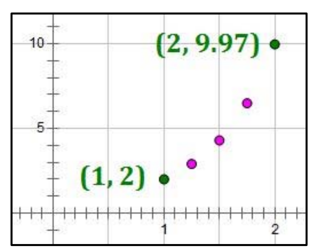
In the corrector step, the estimate of $y'(x_{k+1})$ is based on the value of y_{k+1} generated in the predictor step.

Example 10.7: Let $\frac{dy}{dx} = y'(x) = 2y - x$. Estimate $y(2)$ using 4 time steps if we know $(1, 2)$ is a point on the curve.

We start at point $(x_0, y_0) = (1,2)$, using a time step of $h = \frac{2-1}{4} = 0.25$. The following table shows the iterations required to estimate $y(2)$. Values in the table are rounded to 2 decimals for display, but the exact values are used in all calculations.

k	x_k	y_k	$y'(x_k)$ or $y'(x_{k+1})$	y_{k+1}
0	1.00	2.00	$2\,(2.00) - 1.00 = 3.00$	$2.00 + 3.00\,(0.25) = 2.75$
		Corrector	$2\,(2.75) - 1.25 = 4.25$	$2.00 + (3.00 + 4.25)/2 \cdot (0.25) = 2.91$
1	1.25	2.91	$2\,(2.91) - 1.25 = 4.56$	$2.91 + 4.56\,(0.25) = 4.05$
		Corrector	$2\,(4.05) - 1.50 = 6.59$	$2.91 + (4.56 + 6.59)/2 \cdot (0.25) = 4.30$
2	1.50	4.30	$2\,(4.30) - 1.50 = 7.10$	$4.30 + 7.10\,(0.25) = 6.08$
		Corrector	$2\,(6.08) - 1.75 = 10.40$	$4.30 + (7.10 + 10.40)/2 \cdot (0.25) = 6.49$
3	1.75	6.49	$2\,(6.49) - 1.75 = 11.23$	$6.49 + 11.23\,(0.25) = 9.30$
		Corrector	$2\,(9.30) - 2.00 = 16.59$	$6.49 + (11.23 + 16.59)/2 \cdot (0.25) = 9.97$
4	2.00	9.97		

A plot of successive values of x_k is shown in the graph at right.



The Modified Euler's Method is more complex than Euler's Method, but it tends to be more accurate because it uses a better estimate of the slope in each interval. Though complex, this method is also relatively easy to adapt to a spreadsheet program such as Microsoft Excel.

Order: A numerical method is said to be of order n if it produces exact results for polynomials of degree n or less. Euler's method is of order 1. Modified Euler's Method is of order 2. The Runge-Kutta Method, described on the next page, is of order 4.

Runge-Kutta Method

Runge-Kutta Method an order 4 numerical method for estimating points on a curve using an initial point and slopes of the curve at various locations. Using similar notation to that on the previous pages, the Runge-Kutta Method uses the following formulas:

Note: Since k-values have a specific meaning in this method, we have switched our index variable from k to n.

$$x_{n+1} = x_n + h \qquad\qquad y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where the following k-values are weighted together to obtain incremental values of y.

- $f(x,y)$ is the derivative of the function at x, i.e., $f(x,y) = y'(x)$.
- $k_1 = h \cdot f(x_n, y_n)$
- $k_2 = h \cdot f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1)$
- $k_3 = h \cdot f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2)$
- $k_4 = h \cdot f(x_n + h, y_n + k_3)$

Note that the slope, $f(x,y)$, used in defining each successive k value builds on the slope determined in the previous k value.

Example 10.8: Let $\frac{dy}{dx} = y'(x) = 2y - x$. Estimate $y(2)$ using 4 time steps if we know $(1,2)$ is a point on the curve.

Time Step 1: Once again, we start at point $(x_0, y_0) = (1,2)$, and $h = 0.25$. The following steps show the calculation of $y(1.25)$:

$$(x_0, y_0) = (1,2) \qquad\qquad y' = f(x,y) = 2y - x$$

$$k_1 = h \cdot f(x_0, y_0) = (0.25)f(1,2) = (0.25)(2 \cdot 2 - 1) = 0.75$$

$$\begin{aligned} k_2 = h \cdot f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) &= (0.25)f(1.125, 2.375) \\ &= (0.25)(2 \cdot 2.375 - 1.125) = 0.90625 \end{aligned}$$

$$\begin{aligned} k_3 = h \cdot f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) &= (0.25)f(1.125, 2.453125) \\ &= (0.25)(2 \cdot 2.453125 - 1.125) = 0.9453125 \end{aligned}$$

$$\begin{aligned} k_4 = h \cdot f(x_0 + h, y_0 + k_3) &= (0.25)f(1.25, 2.9453125) \\ &= (0.25)(2 \cdot 2.9453125 - 1.25) = 1.16015625 \end{aligned}$$

$$\begin{aligned} y(1.25) = y_1 &= y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ &= 2 + \frac{1}{6}(0.75 + 2 \cdot 0.90625 + 2 \cdot 0.9453125 + 1.16015625) = \mathbf{2.94} \end{aligned}$$

Time Steps 2 to 4: Performing the same set of calculations for three more steps gives the following values, all rounded to two decimals:

$$y(1.50) = 4.40 \qquad y(1.75) = 6.72 \qquad y(2.00) = 10.48$$

To nine decimal places, with 4 time steps, our calculated value of $y(2.00)$ is 10.479962905. Changing the number of time steps produces the results in the following table.

Number of Time Steps	Value of $y(2.00)$
4	10.479962905
10	10.486111552
20	10.486305959
50	10.486319742
100	10.486320099
200	10.486320122
500	10.486320124
Actual	10.486320124

Notice how the increasing the number of time steps in the calculation improves the accuracy of the results. With 500 time steps the result is accurate to 9 decimal places.

In summary, let's compare the results under the three methods above to the true values for the function defined by our conditions: $y = \left(\frac{5}{4e^2}e^{2x} + \frac{1}{2}x + \frac{1}{4}\right)$.

Estimates of y at Each Time Step Under Four Numerical Methods					
Time Step	x-value	Euler's Method	Modified Euler's Method	Runge-Kutta (4-steps)	Actual Value
1	1.25	2.75	2.90625	2.935546875	2.935901588
2	1.50	3.8125	4.30078125	4.396682739	4.397852286
3	1.75	5.34375	6.488769531	6.724219203	6.727111338
4	2.00	7.578125	9.966125488	10.479962905	10.486320124

Clearly, the higher the order, the more accurate the estimates were for the function defined in the example. This will tend to be true, but will not be true in every case. Increasing the number of steps, and correspondingly decreasing the value of h, will also tend to increase the accuracy of the estimates.

Even though there are a significant number of steps and calculations involved in developing Runge-Kutta estimates, their accuracy may warrant the effort, especially if a spreadsheet proram is readily available to the student.

Vectors

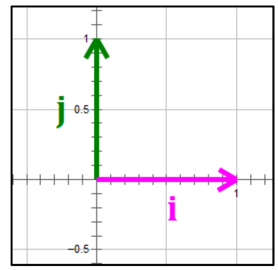
A vector is a quantity that has both magnitude and direction. An example would be wind blowing toward the east at 30 miles per hour. Another example would be the force of 10 kg weight being pulled toward the earth (a force you can feel if you are holding the weight).

Special Unit Vectors

We define unit vectors to be vectors of length 1. Unit vectors having the direction of the positive axes will be quite useful to us. They are described in the chart and graphic below.

Unit Vector	Direction
$\mathbf{i}$	positive x-axis
$\mathbf{j}$	positive y-axis
$\mathbf{k}$	positive z-axis

Graphical representation of unit vectors $\mathbf{i}$ and $\mathbf{j}$ in two dimensions.



Vector Components

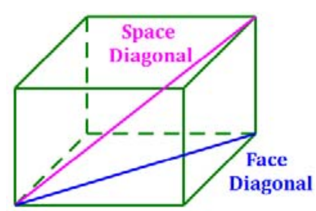
The length of a vector, $\mathbf{v}$, is called its magnitude and is represented by the symbol $\|\mathbf{v}\|$. If a vector's initial point (starting position) is (x_1, y_1, z_1), and its terminal point (ending position) is (x_2, y_2, z_2), then the vector displaces $\boldsymbol{a = x_2 - x_1}$ in the x-direction, $\boldsymbol{b = y_2 - y_1}$ in the y-direction, and $\boldsymbol{c = z_2 - z_1}$ in the z-direction. We can, then, represent the vector as follows:

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

The magnitude of the vector, $\mathbf{v}$, is calculated as:

$$\|\mathbf{v}\| = \sqrt{a^2 + b^2 + c^2}$$

If this looks familiar, it should. The magnitude of a vector in three dimesnsions is determined as the length of the space diagonal of a rectangular prism with sides a, b and c.



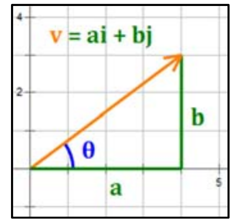
In two dimensions, these concepts contract to the following:

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} \qquad\qquad \|\mathbf{v}\| = \sqrt{a^2 + b^2}$$

In two dimensions, the magnitude of the vector is the length of the hypotenuse of a right triangle with sides a and b.

Vector Properties

Vectors have a number of nice properties that make working with them both useful and relatively simple. Let m and n be scalars, and let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Then,



- If $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$, then $a = \|\mathbf{v}\| \cos\theta$ and $b = \|\mathbf{v}\| \sin\theta$
- Then, $\mathbf{v} = \|\mathbf{v}\| \cos\theta \ \mathbf{i} + \|\mathbf{v}\| \sin\theta \ \mathbf{j}$ (note: this formula is used in Force calculations)
- If $\mathbf{u} = a_1\mathbf{i} + b_1\mathbf{j}$ and $\mathbf{v} = a_2\mathbf{i} + b_2\mathbf{j}$, then $\mathbf{u} + \mathbf{v} = (a_1 + a_2)\mathbf{i} + (b_1 + b_2)\mathbf{j}$
- If $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$, then $m\mathbf{v} = (ma)\mathbf{i} + (mb)\mathbf{j}$
- Define $\mathbf{0}$ to be the zero vector (i.e., it has zero length, so that $a = b = 0$). Note: the zero vector is also called the null vector.

Note: $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ can also be shown with the following notation: $\mathbf{v} = \langle a, b\rangle$. This notation is useful in calculating dot products and performing operations with vectors.

Properties of Vectors

$\mathbf{0} + \mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{v}$	Additive Identity
$\mathbf{v} + (-\mathbf{v}) = (-\mathbf{v}) + \mathbf{v} = \mathbf{0}$	Additive Inverse
$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutative Property
$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	Associative Property
$m(n\boldsymbol{u}) = (mn)\boldsymbol{u}$	Associative Property
$m(\boldsymbol{u} + \boldsymbol{v}) = m\boldsymbol{u} + m\boldsymbol{v}$	Distributive Property
$(m + n)\boldsymbol{u} = m\boldsymbol{u} + n\boldsymbol{u}$	Distributive Property
$1(\mathbf{v}) = \mathbf{v}$	Multiplicative Identity

Also, note that:

$\|m\mathbf{v}\| = \lvert m\rvert \ \|\mathbf{v}\|$	Magnitude Property
$\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$	Unit vector in the direction of $\mathbf{v}$

Vector Dot Product

The Dot Product of two vectors, $\mathbf{u} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{v} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$, is defined as follows:

$$\mathbf{u} \cdot \mathbf{v} = (a_1 \cdot a_2) + (b_1 \cdot b_2) + (c_1 \cdot c_2)$$

It is important to note that the dot product is a scalar, not a vector. It describes something about the relationship between two vectors, but is not a vector itself. A useful approach to calculating the dot product of two vectors is illustrated here:

$$\left.\begin{array}{l} \mathbf{u} = a_1\mathbf{i} + b_1\mathbf{j} + \mathbf{c_1}\mathbf{k} = \langle a_1, b_1, c_1 \rangle \\ \mathbf{v} = a_2\mathbf{i} + b_2\mathbf{j} + \mathbf{c_2}\mathbf{k} = \langle a_2, b_2, c_2 \rangle \end{array}\right\} \text{alternative vector notation}$$

In the example at right the vectors are lined up vertically. The numbers in the each column are multiplied and the results are added to get the dot product. In the example, $\langle 4, -3, 2 \rangle \circ \langle 2, -2, 5 \rangle = 8 + 6 + 10 = 24$.

General	Example 11.1
$\langle a_1, b_1, c_1 \rangle$	$\langle 4, -3, 2 \rangle$
$\circ \langle a_2, b_2, c_2 \rangle$	$\circ \langle 2, -2, 5 \rangle$
$a_1a_2 + b_1b_2 + c_1c_2$	$8 + 6 + 10$ $= 24$

Properties of the Dot Product

Let m be a scalar, and let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Then,

- $\mathbf{0} \circ \mathbf{u} = \mathbf{u} \circ \mathbf{0} = 0$ — Zero Property
- $\mathbf{i} \circ \mathbf{j} = \mathbf{j} \circ \mathbf{k} = \mathbf{k} \circ \mathbf{i} = 0$ — $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are orthogonal to each other.
- $\mathbf{u} \circ \mathbf{v} = \mathbf{v} \circ \mathbf{u}$ — Commutative Property
- $\mathbf{u} \circ \mathbf{u} = \|\mathbf{u}\|^2$ — Magnitude Square Property
- $\mathbf{u} \circ (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \circ \mathbf{v}) + (\mathbf{u} \circ \mathbf{w})$ — Distributive Property
- $m(\mathbf{u} \circ \mathbf{v}) = (m\boldsymbol{u}) \circ \mathbf{v} = \boldsymbol{u} \circ (m\mathbf{v})$ — Multiplication by a Scalar Property

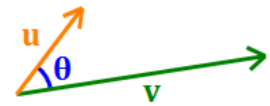
More properties:

- If $\mathbf{u} \circ \mathbf{v} = 0$ and $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal (perpendicular).
- If there is a scalar m such that $m\boldsymbol{u} = \mathbf{v}$, then $\mathbf{u}$ and $\mathbf{v}$ are parallel.
- If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then $\cos\theta = \dfrac{\mathbf{u} \circ \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$

Vector Cross Product

Cross Product

In three dimensions,



Let: $\mathbf{u} = \mathrm{u}_1\mathbf{i} + \mathrm{u}_2\mathbf{j} + \mathrm{u}_3\mathbf{k}$ and $\mathbf{v} = \mathrm{v}_1\mathbf{i} + \mathrm{v}_2\mathbf{j} + \mathrm{v}_3\mathbf{k}$

Then, the Cross Product is given by:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \mathrm{u}_1 & \mathrm{u}_2 & \mathrm{u}_3 \\ \mathrm{v}_1 & \mathrm{v}_2 & \mathrm{v}_3 \end{vmatrix} = (\mathrm{u}_2\mathrm{v}_3 - \mathrm{u}_3\mathrm{v}_2)\,\mathbf{i} + (\mathrm{u}_3\mathrm{v}_1 - \mathrm{u}_1\mathrm{v}_3)\,\mathbf{j} + (\mathrm{u}_1\mathrm{v}_2 - \mathrm{u}_2\mathrm{v}_1)\,\mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\,\sin\theta\;\mathbf{n}$$

The cross product of two nonzero vectors in three dimensions produces a third vector that is orthogonal to each of the first two. This resulting vector $\mathbf{u} \times \mathbf{v}$ is, therefore, normal to the plane containing the first two vectors (assuming $\mathbf{u}$ and $\mathbf{v}$ are not parallel). In the second formula above, $\mathbf{n}$ is the unit vector normal to the plane containing the first two vectors. Its orientation (direction) is determined using the right hand rule.

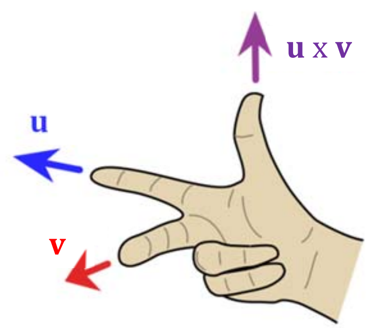
Right Hand Rule

Using your right hand:

- Point your forefinger in the direction of $\mathbf{u}$, and
- Point your middle finger in the direction of $\mathbf{v}$.

Then:

- Your thumb will point in the direction of $\mathbf{u} \times \mathbf{v}$.



In two dimensions,

Let: $\mathbf{u} = \mathrm{u}_1\mathbf{i} + \mathrm{u}_2\mathbf{j}$ and $\mathbf{v} = \mathrm{v}_1\mathbf{i} + \mathrm{v}_2\mathbf{j}$

Then, $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathrm{u}_1 & \mathrm{u}_2 \\ \mathrm{v}_1 & \mathrm{v}_2 \end{vmatrix} = (\mathrm{u}_1\mathrm{v}_2 - \mathrm{u}_2\mathrm{v}_1)$ which is a scalar (in two dimensions).

The cross product of two nonzero vectors in two dimensions is zero if the vectors are parallel. That is, vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if $\mathbf{u} \times \mathbf{v} = 0$.

The area of a parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides and angle θ between them:

$$Area = \mathbf{u} \times \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\,\sin\theta.$$

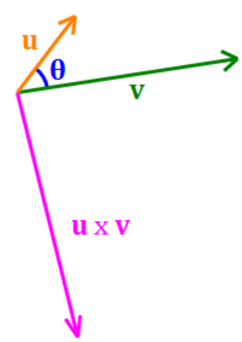
Properties of the Cross Product

Let m be a scalar, and let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Then,

- $\mathbf{0} \times \mathbf{u} = \mathbf{u} \times \mathbf{0} = \mathbf{0}$ — Zero Property
- $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$ — $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are orthogonal to each other
- $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$, $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$ — Reverse orientation orthogonality
- $\mathbf{u} \times \mathbf{u} = \mathbf{0}$ — Every non-zero vector is parallel to itself
- $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$ — Anti-commutative Property
- $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$ — Distributive Property
- $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$ — Distributive Property
- $(\mathrm{m}\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (\mathrm{m}\mathbf{v}) = \mathrm{m}(\mathbf{u} \times \mathbf{v})$ — Scalar Multiplication

More properties:

- If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, then $\mathbf{u}$ and $\mathbf{v}$ are parallel.
- If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then
  - $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\ \|\mathbf{v}\|\ \sin\theta$
  - $\sin\theta = \dfrac{\|\mathbf{u} \times \mathbf{v}\|}{\|\mathbf{u}\|\ \|\mathbf{v}\|}$



Vector Triple Products

Scalar Triple Product

Let: $\mathbf{u} = \mathrm{u}_1\mathbf{i} + \mathrm{u}_2\mathbf{j} + \mathrm{u}_3\mathbf{k}.$ Then the triple product $\mathbf{u} \circ (\mathbf{v} \times \mathbf{w})$ gives a scalar representing the volume of a parallelepiped with $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as edges:

$$\mathbf{u} \circ (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \mathrm{u}_1 & \mathrm{u}_2 & \mathrm{u}_3 \\ \mathrm{v}_1 & \mathrm{v}_2 & \mathrm{v}_3 \\ \mathrm{w}_1 & \mathrm{w}_2 & \mathrm{w}_3 \end{vmatrix}$$

$$\mathbf{u} \circ (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \circ \mathbf{w}$$

Other Triple Products

$\mathbf{u} \circ (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \circ (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$ Duplicating a vector results in a product of $\mathbf{0}$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \circ \mathbf{w})\,\mathbf{v} - (\mathbf{u} \circ \mathbf{v})\,\mathbf{w}$$

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \circ \mathbf{w})\,\mathbf{v} - (\mathbf{v} \circ \mathbf{w})\,\mathbf{u}$$

$$\mathbf{u} \circ (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \circ (\mathbf{w} \times \mathbf{x}) = \mathbf{w} \circ (\mathbf{x} \times \mathbf{v})$$

Note: vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are coplanar if and only if $\mathbf{u} \circ (\mathbf{v} \times \mathbf{w}) = 0$.

No Associative Property

The associative property of real numbers does not translate to triple products. In particular,

$(\mathbf{u} \circ \mathbf{v}) \cdot \mathbf{w} \neq \mathbf{u} \cdot (\mathbf{v} \circ \mathbf{w})$ No associative property of dot products/multiplication

$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ No associative property of cross products

Kinematics (Particle Motion) – Vectors

This page is an extension of the Kinematics pages in Chapter 3, adapted to 3-dimensional space. The corresponding application to 2-dimensional space would remove the third (i.e., z) component of each vector presented. On this page, $\langle\ \rangle$ notation is used for the vectors rather than $\mathbf{i}, \mathbf{j}, \mathbf{k}$ notation.

Position

Position is the location of a particle at a point in time. It may be represented by the vector $\mathbf{s} = \langle x(t), y(t), z(t) \rangle$.

Velocity

Velocity measures the rate of change in position. Instantaneous velocity is the vector of first derivatives of the position vector $\mathbf{v} = \langle x'(t), y'(t), z'(t) \rangle$. Velocity vector components may be either positive or negative.

Speed

Speed is the magnitude of the velocity vector; it is always positive. The formula for speed is:

$$\|\mathbf{v}\| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}$$

Acceleration

Acceleration measures the rate of change in velocity. Instantaneous acceleration is the vector of second derivatives of the position vector $\mathbf{a} = \langle x''(t), y''(t), z''(t) \rangle$.

Moving Among Vectors

The following diagram describes how to move back and forth among the position, velocity and acceleration vectors.

$$Position \quad \underset{\xleftarrow[Integrate]{}}{\xrightarrow{Differentiate}} \quad Velocity \quad \underset{\xleftarrow[Integrate]{}}{\xrightarrow{Differentiate}} \quad Acceleration$$

Displacement

Displacement is a measure of the distance between a particle's starting position and its ending position. The displacement vector from $t = a$ to $t = b$ may be calculated as:

$$\Delta\mathbf{s} = \langle \int_a^b x'(t)dt\,, \int_a^b y'(t)dt\,, \int_a^b z'(t)dt \rangle$$

Gradient

Scalar Fields and Vector Fields

A Scalar Field in three dimensions provides a value at each point in space. For example, we can measure the temperature at each point within an object. The temperature can be expressed as $T = \phi(x, y, z)$. (note: ϕ is the Greek letter phi, corresponding to the English letter "f".)

A Vector Field in three dimensions provides a vector at each point in space. For example, we can measure a magnetic field (magnitude and direction of the magnetic force) at each point in space around a charged particle. The magnetic field can be expressed as $\vec{M} = \vec{V}(x, y, z)$. Note that the half-arrows over the letters M and V indicate that the function generates a vector field.

Del Operator

When looking a scalar field it is often useful to know the rates of change (i.e., slopes) at each point in the x-, y- and z-directions. To obtain this information, we use the Del Operator:

$$\boldsymbol{\nabla} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

Gradient

The Gradient of a scalar field ϕ describes the rates of change in the x, y and z directions at each point in the field in vector form. Therefore, the gradient generates a vector field from the points in the scalar field. The gradient is obtained by applying the del operator to ϕ.

$$grad\ \phi = \boldsymbol{\nabla}\phi = \mathbf{i}\frac{\partial \phi}{\partial x} + \mathbf{j}\frac{\partial \phi}{\partial y} + \mathbf{k}\frac{\partial \phi}{\partial z}$$

$\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}$ and $\frac{\partial \phi}{\partial z}$ are called directional derivatives of the scalar field ϕ.

Example 11.2:

Suppose: $\phi(x, y, z) = \sin x + \ln y + e^{-z}$

Then: $\frac{\partial \phi}{\partial x} = \cos x$, $\frac{\partial \phi}{\partial y} = \frac{1}{y}$ and $\frac{\partial \phi}{\partial z} = -e^{-z}$

So, $\boldsymbol{\nabla}\phi = \mathbf{i}\cos x + \mathbf{j}\frac{1}{y} - \mathbf{k}e^{-z}$; providing all three directional derivatives in a single vector. Over a set of points in space, this results in a vector field.

At point $P = (2, 0.5, -1)$, $\boldsymbol{\nabla}\phi = (\cos 2)\,\mathbf{i} + 2\,\mathbf{j} - e\,\mathbf{k} \sim -0.416\,\mathbf{i} + 2\,\mathbf{j} - 2.718\,\mathbf{k}$

Divergence

Divergence

The Divergence of a vector field describes the flow of material, like water or electrical charge, away from (if positive) or into (if negative) each point in space. The divergence maps the vector at each point in the material to a scalar at that same point (i.e., the dot product of the vector in $\mathbf{V}$ and its associated rates of change in the x, y and z directions), thereby producing a scalar field.

Let $\mathbf{V} = \mathbf{i}\mathrm{V}_x + \mathbf{j}\mathrm{V}_y + \mathbf{k}\mathrm{V}_z$ where $\mathrm{V}_x, \mathrm{V}_y, \mathrm{V}_z$ are each functions in x, y and z. Then,

$$div\,\mathbf{V} = \nabla \circ \mathbf{V} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \circ \left(\mathbf{i}\mathrm{V}_x + \mathbf{j}\mathrm{V}_y + \mathbf{k}\mathrm{V}_z\right)$$

$$= \frac{\partial \mathrm{V}_x}{\partial x} + \frac{\partial \mathrm{V}_y}{\partial y} + \frac{\partial \mathrm{V}_z}{\partial z}$$

Points of positive divergence are referred to as sources, while points of negative divergence are referred to as sinks. The divergence at each point is the net outflow of material at that point, so that if there is both inflow and outflow at a point, these flows are netted in determining the divergence (net outflow) at the point.

Example 11.3:

Let's start with the vector field created by taking the gradient of ϕ on the prior page. Let:

$$\mathbf{V} = \mathbf{i}\cos x + \mathbf{j}\frac{1}{y} - \mathbf{k}e^{-z}$$

In this expression, notice that: $\mathrm{V}_x = \cos x$, $\mathrm{V}_y = \frac{1}{y}$, and $\mathrm{V}_z = -e^{-z}$. Then:

$$div\,\mathbf{V} = \nabla \circ \mathbf{V} = \frac{\partial \mathrm{V}_x}{\partial x} + \frac{\partial \mathrm{V}_y}{\partial y} + \frac{\partial \mathrm{V}_z}{\partial z} = -\sin x - \frac{1}{y^2} + e^{-z}$$

Let's find the value of the divergence at a couple of points, and see what it tells us.

At $P_1 = (-1, 1, 0)$, we have: $div\,\mathbf{v} = -\sin(-1) - \frac{1}{1^2} + e^{-0} = 0.841.$ This value is greater than zero, indicating that P_1 is a "source", and that the vector $\mathbf{v}$ at P_1 produces an outflow.

At $P_2 = (3, -1, 2)$, we have: $div\,\mathbf{v} = -\sin(3) - \frac{1}{(-1)^2} + e^{-2} = -1.006.$ This value is less than zero, indicating that P_2 is a "sink", and that the vector $\mathbf{v}$ at P_2 produces an inflow.

Curl

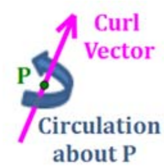
Curl

The Curl of a vector field describes the circulation of material, like water or electrical charge, about each point in the material. The curl maps the vector at each point in the original vector field to another vector (i.e., the cross product of the original vector and its associated rates of change in the x, y and z directions) at that same point, thereby producing a new vector field.

$$curl\,\mathbf{V} \;=\; \boldsymbol{\nabla}\,\mathrm{x}\,\mathbf{V} \;=\; \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\mathrm{x}\left(\mathbf{i}\mathrm{V}_x + \mathbf{j}\mathrm{V}_y + \mathbf{k}\mathrm{V}_z\right)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \mathrm{V}_x & \mathrm{V}_y & \mathrm{V}_z \end{vmatrix} = \mathbf{i}\left(\frac{\partial \mathrm{V}_z}{\partial y} - \frac{\partial \mathrm{V}_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial \mathrm{V}_x}{\partial z} - \frac{\partial \mathrm{V}_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial \mathrm{V}_y}{\partial x} - \frac{\partial \mathrm{V}_x}{\partial y}\right)$$

The curl gives the direction of the axis of circulation of material at a point P. The magnitude of the curl gives the strength of the circulation. If the curl at a point is equal to the zero vector (i.e., $\mathbf{0}$), its magnitude is zero and the material is said to be irrotational at that point.



Example 11.4:

We need to use a more complex vector field for the curl to produce meaningful results. Let:

$$\mathbf{V} = \mathbf{i}\,(yz\cos x) \;+\; \mathbf{j}\left(\frac{xy}{z}\right) \;-\; \mathbf{k}(e^{-xyz})$$

In this expression, notice that: $\mathrm{V}_x = yz\cos x$, $\mathrm{V}_y = \dfrac{xy}{z}$, and $\mathrm{V}_z = -e^{-xyz}$. Then:

$$curl\,\mathbf{V} \;=\; \boldsymbol{\nabla}\,\mathrm{x}\,\mathbf{V} \;=\; \mathbf{i}\left(\frac{\partial \mathrm{V}_z}{\partial y} - \frac{\partial \mathrm{V}_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial \mathrm{V}_x}{\partial z} - \frac{\partial \mathrm{V}_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial \mathrm{V}_y}{\partial x} - \frac{\partial \mathrm{V}_x}{\partial y}\right)$$

$$= \mathbf{i}\left(xze^{-xyz} + \frac{xy}{z^2}\right) + \mathbf{j}\left(y\cos x - yze^{-xyz}\right) + \mathbf{k}\left(\frac{y}{z} - z\cos x\right)$$

Let's find the value of the curl at a point, and see what it tells us. Let $P = (-1, 1, 2)$. Then,

$$curl\,\mathbf{v} = (-2e^2 - 0.25)\,\mathbf{i} + (\cos[-1] - 2e^2)\,\mathbf{j} + (0.5 - 2\cos[-1])\,\mathbf{k} \sim -15.0\mathbf{i} - 14.2\mathbf{j} - 0.6\mathbf{k}$$

The circulation, then, at Point P is around an axis in the direction of: $-15.0\mathbf{i} - 14.2\mathbf{j} - 0.6\mathbf{k}$

The strength of the circulation is given by the magnitude of the curl:

$$\|curl\,\mathbf{v}\| = \sqrt{(-15.0)^2 + (-14.2)^2 + (-0.6)^2} = 20.7$$

Laplacian

Laplacian

The Laplacian Operator is similar to the Del Operator, but involves second partial derivatives.

$$\boldsymbol{\nabla}^2 = \mathbf{i}\frac{\partial^2}{\partial x^2} + \mathbf{j}\frac{\partial^2}{\partial y^2} + \mathbf{k}\frac{\partial^2}{\partial z^2}$$

The Laplacian of a scalar field ϕ is the divergence of the gradient of the field. It is used extensively in the sciences.

$$\boldsymbol{\nabla}^2\phi = \boldsymbol{\nabla} \circ \boldsymbol{\nabla}\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

Example 11.5:

For the scalar field $\phi(x, y, z) = \sin x + \ln y + e^{-z}$, we already calculated the Laplacian in the example for divergence above (but we did not call it that). It is repeated here with Laplacian notation for ease of reference.

Gradient:

$$\boldsymbol{\nabla}\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}$$

For the scalar field defined above: $\frac{\partial\phi}{\partial x} = \cos x$, $\frac{\partial\phi}{\partial y} = \frac{1}{y}$ and $\frac{\partial\phi}{\partial z} = -e^{-z}$

So, $\boldsymbol{\nabla}\phi = \mathbf{i}\cos x + \mathbf{j}\frac{1}{y} - \mathbf{k}e^{-z}$

Laplacian (Divergence of the Gradient):

$$\boldsymbol{\nabla}^2\phi = \boldsymbol{\nabla} \circ \boldsymbol{\nabla}\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = -\sin x - \frac{1}{y^2} + e^{-z}$$

Let's then find the value of the Laplacian at a couple of points.

At $P_1 = (-1, 1, 0)$, we have: $\boldsymbol{\nabla}^2\phi = -\sin(-1) - \frac{1}{1^2} + e^{-0} = 0.841.$

At $P_2 = (3, -1, 2)$, we have: $\boldsymbol{\nabla}^2\phi = -\sin(3) - \frac{1}{(-1)^2} + e^{-2} = -1.006.$

Sequences

Definitions

- A **Sequence** is an ordered set of numbers.
- A **Term** is an element in the ordered set of numbers.
- An **Infinite Sequence** has no end. A **Finite Sequence** has a final term.
- An **Explicit Sequence** is one that defines the terms of the sequence based on the number of the term. By convention, the number of the term is usually expressed in terms of the variables n or k. We talk of the n^{th} *term* or the k^{th} *term* of the sequence or series.
- A **Recursive Sequence** is one that defines its terms based on one or more previous terms.

Types of Sequences

A **term** of a sequence is denoted $\boldsymbol{a_n}$ and an entire sequence of terms $\{\boldsymbol{a_n}\}$. Generally (unless otherwise specified), $n = 1$ for the first term of a sequence, $n = 2$ for the second term, etc.

- **Explicit Sequence:** terms of the sequence $\{a_n\}$ are defined by an Explicit Formula.

  Example 12.1: $\left\{\dfrac{2n}{1+n}\right\} = \left\{\dfrac{2}{2}, \dfrac{4}{3}, \dfrac{6}{4}, \dfrac{8}{5}, \dots\right\}$

  Example 12.2: $\left\{\dfrac{1}{n}\right\} = \left\{1, \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \dots\right\}$

  Example 12.3: $\left\{3 \cdot \left(\dfrac{1}{2}\right)^n\right\} = \left\{\dfrac{3}{2}, \dfrac{3}{4}, \dfrac{3}{8}, \dfrac{3}{16}, \dots\right\}$

  Example 12.4: $\{(-1)^n\} = \{-1, +1, -1, +1, \dots\}$

  Example 12.5: $\{B_n\} = \left\{1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \dots\right\}$

  Note: this is the sequence of Bernoulli Numbers; it begins with B_0.

- **Recursive Sequence:** Usually, one or more initial terms are defined with values in a recursive sequence. Each subsequent term is defined in terms of previous terms.

  Example 12.6: $\{f_{n>2} = f_{n-1} + f_{n-2},\ f_1 = f_2 = 1\} = \{1, 1, 2, 3, 5, 8, 13, \dots\}$

  Example 12.7: $\{f_{n>2} = f_{n-2} - f_{n-1},\ f_1 = 3, f_2 = 1\} = \{3, 1, 2, -1, 3, -4, 7, \dots\}$

More Definitions for Sequences

Monotonic Sequence: A sequence is monotonic if its terms are:

- **Non-increasing** (i.e., $a_{n+1} \leq a_n \ \ \forall\, n$), or
- **Non-decreasing** (i.e., $a_{n+1} \geq a_n \ \ \forall\, n$).
- Note that successive terms may be equal, as long as they do not turn around and head back in the direction from whence they came.
- Often, you can determine whether a sequence is monotonic by graphing its terms.

Bounded Sequence: A sequence is **bounded** if it is bounded from above and below.

- A sequence is bounded from above if there is a number M such that $a_n \leq M \ \ \forall\, n$. The least upper bound is called the **Supremum**.
- A sequence is bounded from below if there is a number N such that $a_n \geq N \ \ \forall\, n$. The greatest lower bound is called the **Infimum**.

Theorems about Sequences

Consider the sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$. The following theorems apply:

Squeeze Theorem:

$$\text{If} \quad a_n \leq b_n \leq c_n \quad \forall \quad n > \text{some } N \quad \text{and} \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L\,, \quad \text{then} \quad \lim_{n\to\infty} b_n = L.$$

Absolute Value Theorem:

$$\text{If} \quad \lim_{n\to\infty} |a_n| = 0\,, \quad \text{then} \quad \lim_{n\to\infty} a_n = 0.$$

Bounded Monotonic Sequence Theorem:

If a sequence is bounded and monotonic, then it converges.

Limit of a Sequence

- **Limit**: $\lim_{n \to \infty} a_n = L$. The limit L exists if we can make a_n as close to L as we like by making n sufficiently large.
- **Convergence:** If the limit of the terms $\{a_n\}$ exists, the sequence is said to be convergent.
- **Divergence:** If the limit of the terms $\{a_n\}$ does not exist, the sequence is said to be divergent.
- Limits are determined in the usual manner.
- Usual properties of limits are preserved in sequences (e. g., addition, scalar multiplication, multiplication, division of limits).

Much more about limits is presented in Chapter 1.

Basic Recursive Sequence Theory

Operators and Annihilators

An operator, applied to a sequence, results in a new sequence. The operator, E^n, shifts a sequence n terms to the left. Sequences can also be added or subtracted, as well as multiplied or divided by a scalar (a number).

An annihilator of a sequence is an operator that converts a sequence into the zero sequence, i.e., $\{0\}$. The typical example of this is the operator $(E-2)$ operating on the sequence $\{2^i\} = \{2, 4, 8, 16, \dots\}$.

Examples of the algebra of operators and annihilators:

Example 12.8: $E\{2^i\} = E\{2, 4, 8, 16, \dots\} = \{4, 8, 16, 32, \dots\} = \{2^{i+1}\}$

Example 12.9: $2 \cdot \{2^i\} = 2 \cdot \{2, 4, 8, 16, \dots\} = \{4, 8, 16, 32, \dots\} = \langle 2^{i+1} \rangle$

Example 12.10:
$$(E-2)\{2^i\} = E\{2^i\} - 2\{2^i\}$$
$$= \{2^{i+1}\} - \{2^{i+1}\} = \{0\}$$

Annihilators – Summary

The following table summarizes some sequence forms that are annihilated by elementary operators:

Form of Sequence	Sample Sequence (starting with a_1)	Annihilator
$\{A\}$	$\{3, 3, 3, 3, \dots\} = \{3\}$	$E-1$
$\{An+B\}$	$\{4, 9, 14, 19, \dots\} = \{5n-1\}$	$(E-1)^2$
$\{An^2+Bn+C\}$	$\{6, 9, 16, 27, \dots\} = \{2n^2-3n+7\}$	$(E-1)^3$
$\{A \cdot r^n\}$	$\{6, 12, 24, 48, \dots\} = \{3 \cdot 2^n\}$	$E-r$
$\{(An+B) \cdot r^n\}$	$\{8, 28, 80, 208, \dots\} = \{(3n+1) \cdot 2^n\}$	$(E-r)^2$
$\{A \cdot r^n + B \cdot s^n\},\ r \neq s$	$\{12, 30, 78, 210, \dots\} = \{3 \cdot 2^n + 2 \cdot 3^n\}$	$(E-r)(E-s)$

Example 12.11: Fibonacci Sequence

The Fibonacci Sequence is defined by the initial conditions, $F_0 = 0$ and $F_1 = 1$, and the Recursive Formula: $F_n = F_{n-1} + F_{n-2}$ for $n > 1$. It looks like this, starting with F_0:

$$\{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \ldots\}$$

Let's find the Explicit Formula (i.e., closed form) for the n^{th} term of the Fibonacci sequence. That is, we will use annihilators to convert:

$$F_n = F_{n-1} + F_{n-2} \qquad \text{into} \qquad F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}.$$

where, $\varphi = \frac{1+\sqrt{5}}{2}$, and $\psi = \frac{1-\sqrt{5}}{2}$, given the initial conditions: $F_0 = 0$ and $F_1 = 1$.

Begin with the Fibonacci Recursive Formula:

$$F_n = F_{n-1} + F_{n-2}, \qquad \text{or equivalently,} \qquad F_{n+2} = F_{n+1} + F_n$$

Convert the equation to operator form in order to identify the annihilator:

$$F_{n+2} = F_{n+1} + F_n$$

$$F_{n+2} - F_{n+1} - F_n = 0$$

$$E^2(F_n) - E(F_n) - F_n = 0$$

$$(E^2 - E - 1)F_n = 0$$

The resulting operator, $(E^2 - E - 1)$, is the annihilator.

Set this equal to zero, and solve it using the quadratic formula:

$$E^2 - E - 1 = 0$$

$$E = \frac{1\pm\sqrt{5}}{2}. \quad \text{Let: } \varphi = \frac{1+\sqrt{5}}{2}, \text{ and } \psi = \frac{1-\sqrt{5}}{2}$$

Because φ and ψ are roots of the equation, the annihilator can be expressed in factored form as:

$$E^2 - E - 1 = (E - \varphi)(E - \psi) = 0$$

Using the annihilator table on the previous page, we see that:

$(E - \varphi)$	annihilates	$\{A \cdot \varphi^n\}$
$(E - \psi)$	annihilates	$\{B \cdot \psi^n\}$

Also from the annihilator table, $(E-\varphi)(E-\psi)$ annihilates sequences of the form:

$$\{A\cdot\varphi^n + B\cdot\psi^n\}.$$

It remains to calculate the values of the coefficients A and B. We do this using the initial (i.e., seed) conditions of the Fibonacci Sequence: $F_0 = 0$ and $F_1 = 1,$ as follows:

$$F_0 = 0 \;=\; A\cdot\varphi^0 + B\cdot\psi^0 \;=\; A + B$$

$$F_1 = 1 \;=\; A\cdot\varphi^1 + B\cdot\psi^1 \;=\; A\cdot\varphi + B\cdot\psi$$

From the expression for F_0, we get:

$$B \;=\; -A$$

Substituting this into the expression for F_1, we get:

$$1 \;=\; A\cdot\varphi - A\cdot\psi \;=\; A\cdot(\varphi-\psi)$$

$$= A\cdot\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right)$$

$$= A\cdot\sqrt{5}$$

This results in the following values for the coefficients:

$$A = \frac{1}{\sqrt{5}}\,, \quad B = -\frac{1}{\sqrt{5}},$$

and, finally, the Explicit Formula for the n^{th} term of the Fibonacci Sequence:

$$\boldsymbol{F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}}$$

where, $\varphi = \frac{1+\sqrt{5}}{2}$, and $\psi = \frac{1-\sqrt{5}}{2}$.

Test Values: $F_6 = \frac{\varphi^6 - \psi^6}{\sqrt{5}} = 8$ ✓ $F_9 = \frac{\varphi^9 - \psi^9}{\sqrt{5}} = 34$ ✓

Process for Finding an Explicit Formula

In summary, the following steps are used to convert a sequence from recursive to explicit form.

Step 1: Identify the Recursive Formula (Example 12.12)

$a_n = 3a_{n-1} + 4a_{n-2} - 12a_{n-3}$ with seed values: $a_0 = 0,\ a_1 = 1,\ a_2 = 2$

So, $\{a_n\} = \{0, 1, 2, 10, 26, 94, 266, 862, \ldots\}$

Step 2: Shift the recursive formula so that its lowest subscript is "n"

$E^3(a_n) \Rightarrow a_{n+3} = 3a_{n+2} + 4a_{n+1} - 12a_n$

Step 3: Use Algebra to get all of the "a" terms on one side of the equation

$a_{n+3} - 3a_{n+2} - 4a_{n+1} + 12a_n = 0$

Step 4: Express the equation in Operator Form

$E^3(a_n) - 3E^2(a_n) - 4E(a_n) + 12a_n = 0$

$(E^3 - 3E^2 - 4E + 12)\ a_n = 0$

Step 5: Factor the Operator Expression

$(E-2)(E+2)(E-3)\ a_n = 0$

Step 6: Use the Annihilator Table to develop a general expression for the Explicit Formula

$a_n = A \cdot 2^n + B \cdot (-2)^n + C \cdot 3^n$

Step 7: Use the seed values of the sequence and the formula in Step 6 to create a set of simultaneous equations for the coefficients (e.g., A, B, C)

$n = 0 \Rightarrow 0 = A + B + C$

$n = 1 \Rightarrow 1 = 2A - 2B + 3C$

$n = 2 \Rightarrow 2 = 4A + 4B + 9C$

Step 8: Solve the simultaneous equations for the coefficients

$A = -\frac{1}{4} = -\frac{5}{20}$ $\qquad B = -\frac{3}{20}$ $\qquad C = \frac{2}{5} = \frac{8}{20}$

Step 9: Write the Explicit Formula for the sequence

$$\boldsymbol{a_n = \frac{-5 \cdot 2^n \ - \ 3 \cdot (-2)^n \ + \ 8 \cdot 3^n}{20}}$$

Step 10: Test the Explicit Formula for a couple of values in the sequence

$a_4 = \frac{-5 \cdot 2^4 - 3 \cdot (-2)^4 + 8 \cdot 3^4}{20} = 26$ ✓ $\qquad a_5 = \frac{-5 \cdot 2^5 - 3 \cdot (-2)^5 + 8 \cdot 3^5}{20} = 94$ ✓

Series

Introduction

A **Series** is an ordered summation of a sequence. If $\{a_i\}$ is an infinite sequence, then the associated **infinite series** (or simply **series**) is:

$$S = \sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$$

The **Partial Sum** containing the first n terms of $\{a_i\}$ is:

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$$

A sequence of partial sums can be formed as follows:

$$\{S_n\} = \{S_1, S_s, S_3, S_4, \dots\}$$

Note the following about these formulas:

- The symbol **S** is the capital Greek letter sigma, which translates into English as **S**, appropriate for the operation of **S**ummation.
- The letter $\boldsymbol{k}$ is used as an **index variable** in both formulas. The **initial (minimum) value** of k is shown below the summation sign and the **terminal (maximum) value** of k is shown above the summation sign. Letters other than k may be used; i, j, and n are common.
- When evaluating a series, make sure you review the initial and terminal values of the index variable. Many mistakes are made by assuming values for these instead of using the actual values in the problem.
- The subscript $\boldsymbol{n}$ in $\boldsymbol{S_n}$ (in the partial sum formula) indicates that the summation is performed only through term a_n. This is true whether the formula starts at $k = 0$, $k = 1$, or some other value of k, though alternative notations may be used if properly identified.

Convergence and Divergence

- If the sequence of partial sums $\{S_n\}$ converges to $\boldsymbol{S}$, the series **converges.** Not surprisingly, $\boldsymbol{S}$ is called the **sum of the series.**
- If the sequence of partial sums $\{S_n\}$ diverges, the series **diverges.**

Key Properties of Series (these also hold for partial sums)

Scalar multiplication

$$\sum_{k=1}^{\infty} c \cdot a_k = c \cdot \sum_{k=1}^{\infty} a_k$$

Sum and difference formulas

$$\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$$

$$\sum_{k=1}^{\infty} (a_k - b_k) = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{\infty} b_k$$

Multiplication

In order to multiply series, you must multiply every term in one series by every term in the other series. Although this may seem daunting, there are times when the products of only certain terms are of interest and we find that multiplication of series can be very useful.

n-th Term Convergence Theorems

$$\text{If } \sum_{k=1}^{\infty} a_k \text{ converges, then } \lim_{k\to\infty} a_k = 0.$$

$$\text{If } \lim_{k\to\infty} a_k \neq 0, \text{then } \sum_{k=1}^{\infty} a_k \text{ diverges.}$$

Power Series

A **Power Series** is an infinite series in which each term is expressed as the product of a constant and a power of a binomial term. Generally, a power series is centered about a particular value of x, which we will call x_0 in the following expression:

$$\boldsymbol{f(x) = \sum_{k=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots}$$

Examples of power series are the Taylor and Maclaurin series covered in Chapter 14.

Telescoping Series

A **Telescoping Series** is one whose terms partially cancel, leaving only a limited number of terms in the partial sums. The general form of a telescoping series, and its sum are is:

$$\sum_{k=1}^{\infty}(a_k - a_{k-1}) = a_1 - \lim_{n\to\infty} a_n$$

Convergence: A telescoping series will converge if and only if the limiting term of the series, $\lim\limits_{n\to\infty} a_n$, is a finite value.

Caution: Telescoping series may be deceptive. Always take care with them and make sure you perform the appropriate convergence tests before concluding that the series sums to a particular value.

Example 13.1:

$$\sum_{k=1}^{\infty}\frac{1}{k^2+k} = \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right)$$

Notice the usefulness of the telescoping approach in the case of a rational function that can be expressed as partial fractions. This approach will not work for some rational functions, but not all of them.

The Partial Sums for this example are:

$$S_1 = \left(1 - \frac{1}{2}\right)$$

$$S_2 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) = \left(1 - \frac{1}{3}\right)$$

$$S_3 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = \left(1 - \frac{1}{4}\right)$$

. . .

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{n+1}\right)$$

Then,

$$S = 1 - \lim_{n\to\infty}\frac{1}{n+1} = \mathbf{1}$$

Geometric Series

A **Geometric Series** has the form:

$$S = \sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots$$

If $|r| < 1$, then the series converges to:

$$S = \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

If $|r| \geq 1$, then the series diverges.

Partial Sums

Partial sums have the form:

$$S_0 = a \qquad S_1 = a + ar \qquad S_2 = a + ar + ar^2 \qquad \ldots$$

$$S_n = \sum_{k=0}^{n} ar^k = a + ar + ar^2 + ar^3 + \cdots + ar^n = \frac{a(1-r^{n+1})}{(1-r)}$$

Example 13.2:

$$S = \sum_{k=0}^{\infty} \frac{0.9}{10^k} = \frac{0.9}{1} + \frac{0.9}{10} + \frac{0.9}{100} + \frac{0.9}{1000} + \cdots = 0.999\overline{9}$$

In this geometric series, we have $a = 0.9$ and $r = \frac{1}{10}$. Therefore the series converges to:

$$S = \sum_{k=0}^{\infty} 0.9 \cdot \left(\frac{1}{10}\right)^k = \frac{0.9}{1 - \frac{1}{10}} = 1$$

This proves, therefore, that $0.999\overline{9} = 1$.

Estimating the Value of a Series with Positive Terms

Let the following be true:

- $f(x)$ is a positive, decreasing, continuous function for all values of $x \geq m,\ m > 0$.
- $f(k) = a_k$ for all integer values of $k \geq m$.
- $S = \sum_{k=1}^{\infty} a_k$ is a convergent series with partial sums $S_n = \sum_{k=1}^{n} a_k$.
- The Remainder Term of the sum, after the n-th term, is defined as: $R_n = S - S_n$.

Then,

$$\int_{n+1}^{\infty} f(x)\,dx \quad < \quad R_n \quad < \quad \int_{n}^{\infty} f(x)\,dx$$

And so,

$$S_n + \int_{n+1}^{\infty} f(x)\,dx \quad < \quad S \quad < \quad S_n + \int_{n}^{\infty} f(x)\,dx$$

Riemann Zeta Functions (p-Series)

Definition

The **Riemann Zeta Function** is defined by the equivalent integral and summation forms:

$$\zeta(x)=\frac{1}{\Gamma(x)}\int_0^{\infty}\frac{t^{x-1}}{e^t-1}dt \qquad\qquad \zeta(x)=\sum_{k=1}^{\infty}\frac{1}{k^x}=1+\frac{1}{2^x}+\frac{1}{3^x}+\frac{1}{4^x}+\cdots$$

The summation form of the function is often called a **p-series** (and p replaces x in the formula).

Zeta functions are generally difficult to evaluate from basic principles. An example of how one of the world's greatest mathematicians evaluated $\zeta(2)$ in 1735 is provided later in this chapter.

Positive Even Integers

Values of $\zeta(x)$ for positive even integer values of x in closed form (as rational expressions involving π) have been calculated by mathematicians. The formula for these is:

$$\zeta(x)=\frac{|B_x|(2\pi)^x}{2(x!)}$$ where B_x is the x-th Bernoulli Number.

$$B_0=1 \quad B_1=-\frac{1}{2} \quad B_2=\frac{1}{6} \quad B_3=0 \quad B_4=-\frac{1}{30} \quad B_5=0 \quad B_6=\frac{1}{42} \quad B_7=0 \quad B_8=-\frac{1}{30} \quad B_9=0 \quad B_{10}=\frac{5}{66} \quad B_{11}=0 \quad B_{12}=-\frac{691}{2730}$$

The decimal approximations below were developed from up to 14 million terms of the p-series using the Algebra App available at www.mathguy.us.

Some values of $\zeta(x)$ for positive even integer values of x are:

$$\zeta(2)=\frac{\pi^2}{6}=1.644933966\ldots \qquad \zeta(8)=\frac{\pi^8}{9450}=1.004077356\ldots$$

$$\zeta(4)=\frac{\pi^4}{90}=1.082323233\ldots \qquad \zeta(10)=\frac{\pi^{10}}{93555}=1.000994575\ldots$$

$$\zeta(6)=\frac{\pi^6}{945}=1.017343061\ldots \qquad \zeta(12)=\frac{691\pi^{12}}{638512875}=1.000246086\ldots$$

Positive Odd Integers

Values of $\zeta(x)$ for positive odd integer values of x do not have a general formula, but can be approximated.

$\zeta(1)$ diverges $\qquad \zeta(7)=1.008349277\ldots$

$\zeta(3)=1.202056903\ldots \qquad \zeta(9)=1.002008392\ldots$

$\zeta(5)=1.036927755\ldots \qquad \zeta(11)=1.000494188\ldots$

Analytic Continuation

Consider the following development:

Let:
$$\begin{array}{rl} P = & 1 - 1 + 1 - 1 + 1 - \cdots \\ P = & \quad\; 1 - 1 + 1 - 1 + \cdots \\ \hline 2P = & 1 \end{array}$$

$$P = \frac{1}{2}$$

Next,

Let:
$$\begin{array}{rl} Q = & 1 - 2 + 3 - 4 + 5 - \cdots \\ Q = & \quad\; 1 - 2 + 3 - 4 + \cdots \\ \hline 2Q = & 1 - 1 + 1 - 1 + 1 - \cdots \end{array}$$

$$2Q = P = \frac{1}{2}$$

$$Q = \frac{1}{4}$$

Then,

Let:
$$\begin{array}{rl} S = & \;\; 1 + 2 + 3 + 4 + 5 + 6 + \cdots \\ -Q = & -1 + 2 - 3 + 4 - 5 + 6 - \cdots \\ \hline S - Q = & \qquad 4 + \quad 8 + \quad 12 + \cdots \end{array}$$

$$S - Q = 4S$$

$$-Q = 3S$$

$$S = -\frac{1}{3}Q = -\frac{1}{3} \cdot \frac{1}{4} = -\frac{1}{12}$$

$$1 + 2 + 3 + 4 + 5 + 6 + \cdots = -\frac{1}{12}$$

And the Riemann Zeta Function value?

This result is consistent with the following value of the Riemann Zeta Function:

$$\zeta(-1) = -\frac{1}{12}$$

How is this possible? See the column to the right for an explanation.

Analytic Continuation

The results in the left-hand column are an example of a concept introduced in Complex Analysis (i.e., Calculus of Complex Variables) called Analytic Continuation. Although the results are correct for the value of the function $\zeta(-1)$, *we cannot conclude that:*

$$1 + 2 + 3 + 4 + 5 + 6 + \cdots = -\frac{1}{12}$$

Why? Because *the series does not converge*; therefore, it does not have a value. What does have a value is the function that overlaps the series where the series converges.

For values of $p > 1$, the Zeta Function and the convergent p-series are equal:

$$\zeta(p) = \sum_{k=1}^{\infty} k^{-p} = \sum_{k=1}^{\infty} \frac{1}{k^p}$$

The function also exists (i.e., *continues*) for values of p for which the series diverges. This is Analytic Continuation.

For another example, consider the following function and series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

This series converges only for $-1 < x < 1$. Yet, we can calculate the function value for $x = 2$.

$$\frac{1}{1-2} = -1$$

This does not imply that:

$$1 + 2 + 2^2 + 2^3 + \cdots = -1$$

Again, *the function continues where the series does not.*

Euler's Development of the Value of $\zeta(2)$

Definition

$$\boldsymbol{\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots}$$

This is also a p-series with $p = 2$. A ***p*-Series** is defined as:

$$\boldsymbol{\sum_{k=1}^{\infty} \frac{1}{k^p}}$$

p-series converge for $p > 1$ and diverge for $p \leq 1$.

Euler's development gives us a glimpse of the extent of his genius. See if you agree.

Euler's Development

1. Begin with the Maclaurin Expansion for: $\sin x$.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ \ldots$$

2. We know that we can fit a curve of degree n through any set of $n+1$ points. Euler proposed that we consider the sine function to be a polynomial of infinite degree that goes through the infinite number of points of the function.

   Further, he noted that the zeros of the polynomial are the zeros of the sine function, i.e., $0, \pm\pi, \pm 2\pi, \pm 3\pi, \pm 4\pi \ldots$. So, the polynomial for $\sin x$ is an infinite product that looks like the following, where c is some constant:

$$\sin x = c \cdot x \, (x^2 - \pi^2)(x^2 - 4\pi^2)(x^2 - 9\pi^2)(x^2 - 16\pi^2) \ldots$$

3. Divide each term on the right by a factor that results in 1's before the x's in each term. Change the lead constant to reflect this. Let's call the new lead constant k.

$$\sin x = k \cdot x \, \frac{(x^2 - \pi^2)}{-\pi^2} \cdot \frac{(x^2 - 4\pi^2)}{-4\pi^2} \cdot \frac{(x^2 - 9\pi^2)}{-9\pi^2} \cdot \frac{(x^2 - 16\pi^2)}{-16\pi^2} \ldots$$

$$= k \cdot x \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\left(1 - \frac{x^2}{16\pi^2}\right) \ldots$$

4. Determine the value of k by dividing each side by x and evaluating the result at $x = 0$.

$$\lim_{x\to 0}\frac{\sin x}{x} = k\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\left(1-\frac{x^2}{16\pi^2}\right)\dots$$

Use L'Hospital's Rule on the left side to determine that $\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x}{1} = 1$. Then,

$$1 = k(1)(1)(1)\dots \qquad \text{so,}\ \ k = 1.$$

5. Rewrite the polynomial in Step 3 with $k = 1$.

$$\sin x = \boldsymbol{x}\left(1-\frac{\boldsymbol{x^2}}{\pi^2}\right)\left(1-\frac{\boldsymbol{x^2}}{4\pi^2}\right)\left(1-\frac{\boldsymbol{x^2}}{9\pi^2}\right)\left(1-\frac{\boldsymbol{x^2}}{16\pi^2}\right)\dots$$

6. Let's examine the coefficient of x^3 in the equation in Step 5.

The coefficient of the x^3 term in this product is obtained by multiplying $\boldsymbol{x}$ by the $\boldsymbol{x^2}$ part of one of the other terms and 1's in the rest of the other terms. We sum the result of this across all of the multiplied terms to get the following x^3 term for the equation in Step 5:

$$\left(-\frac{1}{\pi^2}-\frac{1}{4\pi^2}-\frac{1}{9\pi^2}-\frac{1}{16\pi^2}-\cdots\right)x^3$$

7. The x^3 term in Step 1 must be equal to the x^3 term in Step 6, since both represent the x^3 term in an expansion for $\sin x$. Equating the two coefficients of x^3 gives:

$$-\frac{1}{3!} = -\frac{1}{\pi^2}-\frac{1}{4\pi^2}-\frac{1}{9\pi^2}-\frac{1}{16\pi^2}-\cdots$$

8. Multiply both sides of the result in Step 7 by $-\pi^2$ to get:

$$\frac{\pi^2}{6} = \frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots = \zeta(2)$$

So,

$$\boldsymbol{\zeta(2) = \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}}$$

Bernoulli Numbers and the Riemann Zeta Function

Summation Formulas for Powers of Positive Integers

$$S_0(n-1)=\sum_{k=0}^{n-1}1=1+1+\cdots+1=n \qquad =n$$

$$S_1(n-1)=\sum_{k=0}^{n-1}k=1+2+\cdots+(n-1)=\frac{n(n-1)}{2} \qquad =\frac{1}{2}n^2-\frac{1}{2}n$$

$$S_2(n-1)=\sum_{k=0}^{n-1}k^2=1^2+2^2+\cdots+(n-1)^2=\frac{n(n-1)(2n-1)}{6} \qquad =\frac{1}{3}n^3-\frac{1}{2}n^2+\frac{1}{6}n$$

$$S_3(n-1)=\sum_{k=0}^{n-1}k^3=1^3+2^3+\cdots+(n-1)^3=\frac{n^2(n-1)^2}{4} \qquad =\frac{1}{4}n^4-\frac{1}{2}n^3+\frac{1}{4}n^2$$

$$S_4(n-1)=\sum_{k=0}^{n-1}k^4=1^4+2^4+\cdots+(n-1)^4=\frac{n(n-1)(2n-1)(3n^2-3n-1)}{30}=\frac{1}{5}n^5-\frac{1}{2}n^4+\frac{1}{3}n^3-\frac{1}{30}n$$

The coefficients of the "n" terms are called Bernoulli Numbers. A recursive formula for the Bernoulli numbers is:

$$B_n=\sum_{k=0}^{n}\binom{n}{k}B_k \;\Rightarrow\; B_n=B_0+\binom{n}{1}B_1+\binom{n}{2}B_2+\cdots+\binom{n}{n-2}B_{n-2}+\binom{n}{n-1}B_{n-1}+B_n$$

$$0=B_0+\binom{n}{1}B_1+\binom{n}{2}B_2+\cdots+\binom{n}{n-2}B_{n-2}+nB_{n-1}$$

$$B_{n-1}=-\frac{1}{n}\left[B_0+\binom{n}{1}B_1+\binom{n}{2}B_2+\cdots+\binom{n}{n-2}B_{n-2}\right]=-\frac{1}{n}\sum_{k=0}^{n-2}\binom{n}{k}B_k$$

Then, we can calculate successive Bernoulli Numbers, starting with $B_0=1$ as:

$$B_0=1$$

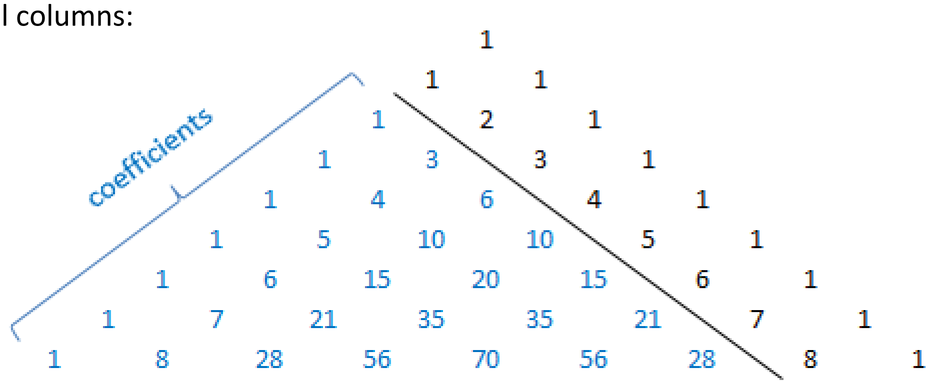
$$B_1=-\frac{1}{2}(1\cdot B_0)=-\frac{1}{2}$$

$$B_2=-\frac{1}{3}(1\cdot B_0+3\cdot B_1)=-\frac{1}{3}\left[1\cdot(1)+3\cdot\left(-\frac{1}{2}\right)\right]=\frac{1}{6}$$

$$B_3=-\frac{1}{4}(1\cdot B_0+4\cdot B_1+6\cdot B_2)=-\frac{1}{4}\left[1\cdot(1)+4\cdot\left(-\frac{1}{2}\right)+6\cdot\left(\frac{1}{6}\right)\right]=0$$

$$B_4=-\frac{1}{5}(1\cdot B_0+5\cdot B_1+10\cdot B_2+10\cdot B_3)=-\frac{1}{5}\left[1\cdot(1)+5\cdot\left(-\frac{1}{2}\right)+10\cdot\left(\frac{1}{6}\right)+10\cdot 0\right]=-\frac{1}{30}$$

The blue numbers in the above formulas are the values from Pascal's Triangle, excluding the last two diagonal columns:



$B_n = 0$ for every odd $n > 1$.

Below are values of B_n for even values of $n \leq 24$. Note: $|B_{2n}| \sim 4\pi\sqrt{e}\left(\frac{n}{\pi e}\right)^{2n+1/2}$ as $n \to \infty$

$B_2 = \frac{1}{6}$	$B_8 = -\frac{1}{30}$	$B_{14} = \frac{7}{6}$	$B_{20} = -\frac{174611}{330}$
$B_4 = -\frac{1}{30}$	$B_{10} = \frac{5}{66}$	$B_{16} = -\frac{3617}{510}$	$B_{22} = \frac{854513}{138}$
$B_6 = \frac{1}{42}$	$B_{12} = -\frac{691}{2730}$	$B_{18} = \frac{43867}{798}$	$B_{24} = -\frac{236364091}{2730}$

Bernoulli Numbers relate to the Riemann Zeta Functions as follows:

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{1}{1^{2n}} + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \cdots = \frac{|B_{2n}|(2\pi)^{2n}}{2\,(2n)!}$$

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} \qquad \zeta(6) = \sum_{k=1}^{\infty} \frac{1}{k^6} = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \cdots = \frac{\pi^6}{945}$$

$$\zeta(4) = \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90} \qquad \zeta(8) = \sum_{k=1}^{\infty} \frac{1}{k^8} = \frac{1}{1^8} + \frac{1}{2^8} + \frac{1}{3^8} + \cdots = \frac{\pi^8}{9450}$$

The function $\frac{x}{e^x - 1}$ expands using Bernoulli Numbers as follows:

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_n \frac{x^k}{k!} = 1 - \frac{1}{2}x + \frac{1}{6}x^2 - \frac{1}{30}x^4 + \frac{1}{42}x^6 - \cdots$$

Series Convergence Tests

Integral Test

Let $\sum a_k$ be a positive series, and

let $f(x)$ be a continuous, positive, decreasing function on $[m, +\infty)$, $m > 0$, such that $f(k) = a_k$ for every $k \geq m$. Then,

$$\sum_{k=m}^{\infty} a_k \text{ converges if and only if } \int_m^{\infty} f(x)\,dx \text{ converges.}$$

If the series converges, $\displaystyle\sum_{k=m}^{\infty} a_k \neq \int_m^{\infty} f(x)\,dx$. That is, the sum of the series and the integral will have different values.

Comparison Test

Let $\sum a_k$ and $\sum b_k$ be positive series. If there is an index m, beyond which $a_k < b_k$ for every $k \geq m$, then:

- If $\sum b_k$ converges, so does $\sum a_k$.
- If $\sum a_k$ diverges, so does $\sum b_k$.

Limit Comparison Test

Let $\sum a_k$ and $\sum b_k$ be positive series such that $0 < \lim\limits_{n\to\infty} \frac{a_n}{b_n} < \infty$. Then:

- $\sum a_k$ converges if and only if $\sum b_k$ converges.
- $\sum a_k$ diverges if and only if $\sum b_k$ diverges.

Absolute and Conditional Convergence

- $\sum a_k$ is *absolutely convergent* if $\sum |a_k|$ is convergent.
- $\sum a_k$ is *conditionally convergent* if it is convergent but not absolutely convergent.

Term Rearrangement

- If an infinite series is *absolutely convergent,* the terms can be rearranged without affecting the resulting sum.
- If an infinite series is *conditionally convergent,* a rearrangement of the terms may affect the resulting sum.

Ratio Test

Let $\sum a_k$ be a series. Then consider the n-th and $(n+1)$-th terms:

If $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| < 1$, then: $\sum a_k$ is absolutely convergent.

If $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| > 1$, then: $\sum a_k$ is divergent.

If $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 1$, then no conclusion about convergence or divergence can be drawn.

Example 13.3:

Determine whether the following series converges or diverges: $\displaystyle\sum_{k=1}^{\infty}\frac{k^k}{k!}$

$$\text{Ratio} = \left|\frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}}\right| = \frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n} = \frac{(n+1)\cdot(n+1)^n}{(n+1)\cdot n!}\cdot\frac{n!}{n^n}$$

$$= \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$$

Then,

$$\lim_{n\to\infty}\left(1+\tfrac{1}{n}\right)^n = e > 1$$

Since $e > 1$, the series diverges.

Root Test

Let $\sum a_k$ be a series. Then consider the n-th term:

$$\text{If } \lim_{n\to\infty} \sqrt[n]{|a_n|} < 1, \text{ then: } \sum a_n \text{ is absolutely convergent.}$$

$$\text{If } \lim_{n\to\infty} \sqrt[n]{|a_n|} > 1, \text{ then: } \sum a_n \text{ is divergent.}$$

$$\text{If } \lim_{n\to\infty} \sqrt[n]{|a_n|} = 1, \text{ then no conclusion about convergence or divergence can be drawn.}$$

Example 13.4:

Determine whether the following series converges or diverges: $\displaystyle\sum_{k=1}^{\infty} \left(\frac{2k+3}{3k+2}\right)^k$

$$\text{Root} = \sqrt[n]{\left|\left(\frac{2n+3}{3n+2}\right)^n\right|} = \sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n} = \frac{2n+3}{3n+2} = \frac{2+\frac{3}{n}}{3+\frac{2}{n}}$$

Then,

$$\lim_{n\to\infty} \left(\frac{2+\frac{3}{n}}{3+\frac{2}{n}}\right) = \frac{2}{3} < 1$$

Since $\frac{2}{3} < 1$, the series converges.

Dirichlet's Convergence Test

Dirichlet's Test

If $\sum_{k=1}^{\infty} a_k$ has bounded partial sums and $\{v_k\}$ is a decreasing positive sequence with $\lim_{k\to\infty} v_k = 0$, then the series $\sum_{k=1}^{\infty}(a_k \cdot v_k)$ converges.

Example 13.5: Prove that the series $\sum_{k=1}^{\infty}\frac{\cos k}{k}$ converges using **Dirichlet's Convergence Test.**

Using the notation shown above, we will let $a_k = \cos k$, and $v_k = \frac{1}{k}$. We require only that $\sum_{k=1}^{\infty}\cos k$ be bounded since $\lim_{k\to\infty}\frac{1}{k}$ clearly decreases to 0.

Start by proving the following Trigonometric identity:

$$\cos k = \frac{\sin\left(k+\frac{1}{2}\right) - \sin\left(k-\frac{1}{2}\right)}{2\sin\frac{1}{2}}$$

$$= \frac{\left(\sin k\cos\frac{1}{2} + \cos k\sin\frac{1}{2}\right) - \left(\sin k\cos\frac{1}{2} - \cos k\sin\frac{1}{2}\right)}{2\sin\frac{1}{2}} = \frac{2\cos k\sin\frac{1}{2}}{2\sin\frac{1}{2}} = \cos k \quad \checkmark$$

Next, let's look at the n-th Partial Sum in light of the above identity. Note that it telescopes:

$$\sum_{k=1}^{n}\cos k = \sum_{k=1}^{n}\frac{\sin\left(k+\frac{1}{2}\right) - \sin\left(k-\frac{1}{2}\right)}{2\sin\frac{1}{2}}$$

$$= \frac{\sin\left(1\frac{1}{2}\right) - \sin\left(\frac{1}{2}\right)}{2\sin\frac{1}{2}} + \frac{\sin\left(2\frac{1}{2}\right) - \sin\left(1\frac{1}{2}\right)}{2\sin\frac{1}{2}} + \cdots + \frac{\sin\left(n+\frac{1}{2}\right) - \sin\left(n-\frac{1}{2}\right)}{2\sin\frac{1}{2}}$$

$$= \frac{\sin\left(n+\frac{1}{2}\right) - \sin\left(\frac{1}{2}\right)}{2\sin\left(\frac{1}{2}\right)}$$

Note that $\sin\left(n+\frac{1}{2}\right)$ is bounded in the range $[-1,1]$. Therefore, $\sum_{k=1}^{n}\cos k$ is bounded in the range: $\left[\frac{-1-\sin\left(\frac{1}{2}\right)}{2\sin\left(\frac{1}{2}\right)}, \frac{1-\sin\left(\frac{1}{2}\right)}{2\sin\left(\frac{1}{2}\right)}\right]$, and so the original series converges.

Abel's Convergence Test

Abel's Test

If $\sum_{k=1}^{\infty} a_k$ converges and $\{v_k\}$ is a monotonic bounded sequence, then the series $\sum_{k=1}^{\infty} (a_k \cdot v_k)$ converges.

Example 13.6: Prove that the series $\sum_{k=1}^{\infty} \frac{\cos\left(\frac{1}{k^2}\right)}{k^2}$ converges using **Abel's Convergence Test.**

Using the notation shown above, we will let $a_k = \frac{1}{k^2}$, and $v_k = \cos\left(\frac{1}{k^2}\right)$. We need to show that a) $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges and b) $\cos\left(\frac{1}{k^2}\right)$ is both monotonic and bounded.

First, identify $\sum_{k=1}^{\infty} \frac{1}{k^2}$ as a p-series, with $p = 2$, so it converges.

Second, let's look at some values of $\cos\left(\frac{1}{k^2}\right)$ in the table to the right. The sequence is clearly monotonic and is bounded by the value of $\cos 0 = 1$.

$$\lim_{k\to\infty} \cos\left(\frac{1}{k^2}\right) = 1$$

We have met both requirements of Abel's Convergence Test, and we can conclude that the given series converges.

Note: the series in this example could also have been determined to be convergent (using the comparison test) by comparing it to a p-series with $p = 2$.

$\boldsymbol{k}$	$\boldsymbol{\cos\left(\frac{1}{k^2}\right)}$
1	$\cos\left(\frac{1}{1}\right) \sim 0.5403$
2	$\cos\left(\frac{1}{4}\right) \sim 0.9689$
3	$\cos\left(\frac{1}{9}\right) \sim 0.9938$
4	$\cos\left(\frac{1}{16}\right) \sim 0.9980$
5	$\cos\left(\frac{1}{25}\right) \sim 0.9992$

Alternating Series

The general form for an Alternating Series that includes an error term is:

$$f(x) = \sum_{k=1}^{\infty} (-1)^{k-1}\, a_k = a_1 - a_2 + a_3 - a_4 + \cdots$$

Theorem: If the sequence, $\langle a_n \rangle$, is positive and non-increasing, and $\lim_{n\to\infty} a_n = 0$,

Then: $\sum (-1)^{k-1} a_k$ converges, and

If R_n is the n^{th} error term, then: $|R_n| < a_{n+1}$

Error Term

The maximum error in a converging alternating series after n terms is term $(-1)^n a_{n+1}$. Using this, we can estimate the value of a series to a desired level of accuracy.

Example 13.7: Approximate the following sum to 4 decimal places: $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{6^k}$

We need to find term $n+1$ to estimate the error. For the series provided, this term is $\frac{(-1)^n}{6^{n+1}}$. The $(-1)^n$ term simply indicates the direction of the error. The magnitude of the error is the balance of the error term, i.e., $\frac{1}{6^{n+1}}$.

In order to find an approximation of the series to 4 decimal places, we need an error less than 0.00005. So, we want:

$$\frac{1}{6^{n+1}} < 0.00005$$

We can solve this using logarithms or by taking successive powers of $\frac{1}{6}$. Either way, we find: $\frac{1}{6^5} \sim 0.0001286 > 0.00005$ and $\frac{1}{6^6} \sim 0.0000214 < 0.00005$, so $n+1=6$, and $n=5$.

Using 5 terms of the alternating series, we find that the value of the sum to 4 decimal places is:

$$\sum_{k=1}^{5} \frac{(-1)^{k-1}}{6^k} = \frac{1}{6} - \frac{1}{36} + \frac{1}{216} - \frac{1}{1296} + \frac{1}{7776} = \mathbf{0.1428755}$$

The actual value of the series is $\frac{1}{7} \sim 0.1428571$, so we can see that the desired level of accuracy has been achieved.

Absolute and Conditional Convergence

- $\sum a_k$ is *absolutely convergent* if $\sum |a_k|$ is convergent.
- $\sum a_k$ is *conditionally convergent* if it is convergent but not absolutely convergent.

Term Rearrangement

- If an infinite series is *absolutely convergent,* the terms can be rearranged without affecting the resulting sum.
- If an infinite series is *conditionally convergent,* a rearrangement of the terms may affect the resulting sum.

More Theorems about Absolutely Convergent Series

The following theorems apply to absolutely convergent series (i.e., absolutely convergent alternating series and convergent series of decreasing positive terms):

- The commutative law applies to terms in an absolutely convergent series; i.e., terms can be rearranged without affecting the value of the series.
- Every sub-series of an absolutely convergent series is absolutely convergent; i.e., terms can be omitted and the result is an absolutely convergent series.
- The sum, difference and product of absolutely convergent series are absolutely convergent. Furthermore, if $\sum a_k$ and $\sum b_k$ are two absolutely convergent series such that $\mathrm{A} = \sum a_k$ and $\mathrm{B} = \sum b_k$, then:
  - $\sum a_k + \sum b_k = \mathrm{A} + \mathrm{B}.$
  - $\sum a_k - \sum b_k = \mathrm{A} - \mathrm{B}.$
  - $\sum a_k \cdot \sum b_k = \mathrm{A} \cdot \mathrm{B}.$

Radius and Interval of Convergence of Power Series

Consider the Power Series:

$$\boldsymbol{f(x) = \sum_{k=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots}$$

Definitions:

- Center: The value a is called the center of the power series. Many power series have a center of $a = 0$.
- Coefficients: The values c_n are called the coefficients of the power series.
- Radius of Convergence: The series may converge for certain values of x and diverge for other values of x. If the series converges for all values of x within a certain distance, R, from a, i.e., for x on the interval $(a - R, a + R)$, we call R the radius of convergence of the series.
- Interval of Convergence: The set of all values of x for which the power series converges is called the interval of convergence of the series. The interval of convergence is closely related to the radius of convergence; it includes the open interval $(a - R, a + R)$, and may also include one or both endpoints of that interval.

Finding the Radius and Interval of Convergence

The radius of convergence is found using the Ratio Test or the Root Test. To find the interval of convergence, the series defined at each endpoint of the interval must be tested separately.

Example 13.8: Consider the power series: $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{k=1}^{\infty} (-1)^k x^k$

Using the Ratio Test, we find:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}x^{n+1}}{(-1)^n x^n}\right| = \lim_{n\to\infty} |x| < 1 \quad \ldots \quad \text{in the open interval: } x \in (-1, 1).$$

So, this series has a radius of convergence: $R = 1$ about a center of $x = 0$.

To find the interval of convergence of the series, we must test the endpoints, i.e., $x = \pm 1$.

When $x = 1$, we get $\frac{1}{1+1} = 1 - 1 + 1^2 - 1^3 + \cdots$, which diverges.

When $x = -1$, we get $\frac{1}{1-1} = 1 + 1 + 1^2 + 1^3 + \cdots$, which also diverges.

The interval of convergence, then, is $(-1, 1)$. It does not include either endpoint.

Differentiating or Integrating Power Series

When differentiating or integrating a Power Series, we differentiate or integrate term-by-term.

Example 13.9: Integrate the power series: $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$

$$\int \frac{1}{1+x}\,dx = \int (1 - x + x^2 - x^3 + \cdots)\,dx$$

$$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots + C$$

The result of the integration turns out to be the power series for $\ln(1+x)$, plus a constant, which we would expect to be the case because:

$$\int \frac{1}{1+x}\,dx = \ln(1+x) + C.$$

Theorem: Differentiation of a Power Series

If a function f is defined by a power series with radius of convergence R, then:

- f is differentiable on the open interval defined by R.
- $f'(x)$ is found by term-by-term differentiation of the power series for f.
- The resulting power series for f' also has radius of convergence R.
- The interval of convergence of f' may be the same as that for f, or it may lose either or both endpoints.

Theorem: Integration of a Power Series

If a function f is defined by a power series with radius of convergence R, then:

- $F(x) = \int f(x)\,dx$ is found by term-by-term integration of the power series for f.
- The resulting power series for F also has radius of convergence R.
- The interval of convergence of F may be the same as that for f, or it may gain either or both endpoints.

Differentiation: $f'(x)$

Term-by-term differentiation.

Has same Ratio of Convergence.

Interval of Convergence may lose one or both endpoints.

Relative to $f(x)$

Integration: $F(x) = \int f(x)\,dx$

Term-by-term integration.

Has same Ratio of Convergence.

Interval of Convergence may gain one or both endpoints.

Example 13.10: The Maclaurin Series for $\frac{1}{1+x}$ is:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

As shown on a previous page, its interval of convergence is $(-1, 1)$.

Integrating term-by-term we get:

$$\int \frac{1}{1+x}\,dx = \int (1 - x + x^2 - x^3 + \cdots)\,dx$$

$$\ln(1+x) + C = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots + C \;=\; C + \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}$$

For the new series, $\ln(1+x) + C$, note that "$+C$" has no impact on whether the series converges or diverges at any point. Then,

Using the Ratio Test, we find:

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n x^{n+1}/(n+1)}{(-1)^{n-1}x^n/n}\right| = \lim_{n\to\infty}\left|x\cdot\frac{n}{n+1}\right| = \lim_{n\to\infty}|x| < 1$ ... in the open interval: $x \in (-1, 1)$.

So, this series also has a radius of convergence $R = 1$ about a center of $x = 0$.

To find the interval of convergence of the series, we must test the endpoints, i.e., $x = \pm 1$.

When $x = 1$, we get $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, which converges by the alternating series test.

When $x = -1$, we get $\ln 0 = -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots$, which diverges (it is the negative harmonic series, and $\ln 0$ is undefined).

The interval of convergence, then, is $(-1, 1]$. It includes the right endpoint.

Conclusion: In the case of this example, the interval of convergence of the integrated series picks up the endpoint at $x = 1$.

McCartin Table: Summary of Basic Tests for Series

Test		Series Form	Conditions for Convergence	Conditions for Divergence	Comments
n-th term (tests for divergence only)		$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} a_n = 0$ required, but not sufficient	$\lim_{n\to\infty} a_n \neq 0$	This test should always be performed first.
Special Series	Telescoping Series	$\sum_{n=1}^{\infty} (a_n - a_{n+1})$	$\lim_{n\to\infty} a_n$ is finite	$\lim_{n\to\infty} a_n$ not finite	$S = a_1 - \lim_{n\to\infty} a_n$
	Geometric Series ($a \neq 0$)	$\sum_{n=1}^{\infty} ar^{n-1}$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	$S = \dfrac{a}{1-r}$
	p-Series	$\sum_{n=1}^{\infty} \dfrac{1}{n^p}$	$p > 1$	$p \leq 1$	$\sum_{n=1}^{\infty} \dfrac{1}{n^p} = \zeta(p)$ (1)
Alternating Series ($0 < a_{n+1} \leq a_n$)		$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$\lim_{n\to\infty} a_n = 0$	$\lim_{n\to\infty} a_n \neq 0$	Remainder: $\lvert R_n\rvert \leq a_{n+1}$
Integral (f is positive, continuous, and decreasing)		$\sum_{n=1}^{\infty} a_n$	$\int_1^{\infty} f(x)\,dx$ converges (2)	$\int_1^{\infty} f(x)\,dx$ diverges	Remainder: $0 < R_n < \int_n^{\infty} f(x)\,dx$
Comparison ($a_n > 0, b_n > 0$)		$\sum_{n=1}^{\infty} a_n$	$a_n \leq b_n$ $(n > m)$ $\sum_{n=1}^{\infty} b_n$ converges	$a_n \geq b_n$ $(n > m)$ $\sum_{n=1}^{\infty} b_n$ diverges	Comparison of a_n and b_n need only exist for n beyond some index m.
Limit Comparison ($a_n > 0, b_n > 0$)		$\sum_{n=1}^{\infty} a_n$	$0 < \lim_{n\to\infty} \dfrac{a_n}{b_n} < \infty$ $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \dfrac{a_n}{b_n} > 0$ $\sum_{n=1}^{\infty} b_n$ diverges	Could use $\lim_{n\to\infty} \frac{b_n}{a_n}$ instead of $\lim_{n\to\infty} \frac{a_n}{b_n}$ in the conditions.
Ratio		$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert\dfrac{a_{n+1}}{a_n}\right\rvert < 1$ (absolute convergence)	$\lim_{n\to\infty} \left\lvert\dfrac{a_{n+1}}{a_n}\right\rvert > 1$	Test inconclusive if: $\lim_{n\to\infty} \left\lvert\dfrac{a_{n+1}}{a_n}\right\rvert = 1.$ Use another test.
Root		$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} < 1$ (absolute convergence)	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} > 1$	Test inconclusive if: $\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = 1.$ Use another test.

Notes: (1) Riemann zeta function. (2) If the series converges, $\sum a_n \neq \int_1^{\infty} f(x)\,dx$.

Taylor and Maclaurin Series

Taylor Series

A Taylor series is an expansion of a function around a given value of x. Generally, it has the following form around the point $x = a$:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Maclaurin Series

A Maclaurin series is a Taylor Series around the value $x = 0$. Generally, it has the following form:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots$$

Example 14. 1:

Find the Maclaurin expansion for $f(x) = e^x$:

$f(x) = e^x \qquad f(0) = e^0 = 1$

$f'(x) = e^x \qquad f'(0) = e^0 = 1$

$f''(x) = e^x \qquad f''(0) = e^0 = 1$

. . .

$f^{(n)}(x) = e^x \qquad f^{(n)}(0) = e^0 = 1$

Substituting these values into the Maclaurin expansion formula (and recalling that $0! = 1$) we get:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Example 14.2:

Find the Maclaurin expansion for $f(x) = \ln(1+x)$:

$$f(x) = \ln(1+x) \qquad f(0) = \ln(1+0) = 0$$

$$f'(x) = \frac{1}{1+x} \qquad f'(0) = \frac{1}{1+0} = 1$$

$$f''(x) = -\frac{1}{(1+x)^2} \qquad f''(0) = -\frac{1}{(1+0)^2} = -1 = -1!$$

$$f'''(x) = \frac{2}{(1+x)^3} \qquad f'''(0) = \frac{2}{(1+0)^3} = 2 = 2!$$

$$f^{iv}(x) = -\frac{6}{(1+x)^4} \qquad f^{iv}(0) = -\frac{6}{(1+x)^4} = -6 = -3!$$

. . .

$$f^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{(1+x)^n} \qquad f^{(n)}(0) = (-1)^{n-1}(n-1)!$$

Substituting these values into the Maclaurin expansion formula, we get:

$$\ln(1+x) = x + \frac{-1}{2!}x^2 + \frac{2!}{3!}x^3 + \frac{-3!}{4!}x^4 + \cdots + \frac{(-1)^{n-1}(n-1)!}{n!}x^n + \cdots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}$$

Taylor Series Convergence Theorem

A Taylor Series for a function $f(x)$ that has derivatives of all orders on an open interval centered at $x = a$ converges if and only if:

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{f^{(n+1)}(x^*)}{(n+1)!}(x-a)^{n+1} = 0$$

The term $R_n(x)$ is called the Lagrange Remainder; x^* is the value of x that produces the greatest value of $f^{(n+1)}(x)$ between a and x. See more on the Lagrange Remainder on the next page.

LaGrange Remainder

The form for a Taylor Series about $x = a$ that includes an error term is:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(x)$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{(n)!}(x-a)^n + R_n(x)$$

The term $R_n(x)$ is called the Lagrange Remainder, and has the form:

$$R_n(x) = \frac{f^{(n+1)}(x^*)}{(n+1)!}(x-a)^{n+1}$$

where x^* produces the greatest value of $f^{(n+1)}(x)$ between a and x.

This form is typically used to approximate the value of a series to a desired level of accuracy.

Example 14.3: Approximate $\sqrt{e}$ using five terms of the Maclaurin Series (i.e., the Taylor Series about $x = 0$) for e^x and estimate the maximum error in the estimate.

Using five terms and letting $x = \frac{1}{2}$, we get:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + R_4(x)$$

$$e^{1/2} \sim 1 + \frac{1}{2} + \frac{\left(\frac{1}{2}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)^3}{3!} + \frac{\left(\frac{1}{2}\right)^4}{4!} = 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + \frac{1}{384} = \mathbf{1.6484375}$$

To find the maximum potential error in this estimate, calculate:

$R_4(x) = \frac{f^{(5)}(x^*)}{5!}x^5$ for $x = \frac{1}{2}$ and x^* between 0 and $\frac{1}{2}$.

Since $f(x) = e^x$, the fifth derivative of f is: $f^{(5)}(x) = e^x$. The maximum value of this between $x = 0$ and $x = \frac{1}{2}$ occurs at $x = \frac{1}{2}$. Then,

$f^{(5)}\left(\frac{1}{2}\right) = e^{1/2} < 1.65$ based on our estimate of 1.6484375 above (we will check this after completing our estimate of the maximum error). Combining all of this,

$$R_4\left(\frac{1}{2}\right) = \frac{f^{(5)}\left(\frac{1}{2}\right)}{5!}\left(\frac{1}{2}\right)^5 < \frac{1.65}{5!}\left(\frac{1}{2}\right)^5 = \mathbf{0.0004297}$$

Note that the maximum value of $\sqrt{e}$, then, is $1.6484375 + 0.0004297 = 1.6488672$, which is less than the 1.65 used in calculating $R_4\left(\frac{1}{2}\right)$, so our estimate is good. The actual value of $\sqrt{e}$ is $1.6487212 \ldots$.

e

What is "*e*"?

- Euler's number, ***e*** is the base of the natural logarithms.
- *e* is a transcendental number, meaning that it is not the root of any polynomial with integer coefficients.

What Makes "*e*" so Special?

e shows up over and over in mathematics, especially in regard to limits, derivatives, and integrals. In particular, it is noteworthy that:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \qquad e = \lim_{n\to\infty}\left(\frac{n}{\sqrt[n]{n!}}\right) \qquad \frac{d}{dx}(e^x) = e^x \qquad \int_1^e \frac{dx}{x} = 1$$

$$\lim_{n\to\infty}\left(1+\frac{1}{bn}\right)^{an} = e^{a/b}$$

Perhaps, most interestingly, the following equation, called **Euler's Equation**, relates five seemingly unrelated mathematical constants to each other.

$$\boldsymbol{e^{i\pi} + 1 = 0}$$

Some Series Representations of e

There are many more series involving e. A sampling of these is provided at: http://mathworld.wolfram.com/e.html.

$$e = \sum_{k=0}^{\infty}\frac{1}{k!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots$$

$$e = \left[\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\right]^{-1} = \frac{1}{1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \cdots}$$

Decimal Expansion

e =2.7 1828 1828 4590 4523 5360 2874 7135 2662 4977 5724 7093 6999 5957 4966 …

The web site http://antwrp.gsfc.nasa.gov/htmltest/gifcity/e.2mil shows the decimal expansion of *e* to over 2 million digits.

Derivation of Euler's Formula by Integration

Start with: $y = \cos x + i \sin x$ [note that $(0, 1)$ is a point on this function]

Then: $dy = (-\sin x + i \cos x)\ dx$

$dy = (i \cos x - \sin x)\ dx$

$dy = iy\ dx$

$\frac{dy}{y} = i\ dx$

Integrate: $\int \frac{dy}{y} = \int i\ dx$

$\ln y = i\,x + C$ [note that $C = 0$ since $(0, 1)$ is a point on this function]

$y = e^{ix}$

Final Result: $\boxed{e^{ix} = \cos x + i \sin x}$

Very Cool Sub-Case

When $x = \pi$, Euler's equation becomes:

$e^{i\pi} = \cos \pi + i \sin \pi$

or, $e^{i\pi} = -1$ Note that this will allow us to calculate logarithms of negative numbers.

Rewriting this provides an equation that relates five of the most important mathematical constants to each other:

$$\boxed{e^{i\pi} + 1 = 0}$$

Derivation of Euler's Formula Using Power Series

A Power Series about zero is an infinite series of the form:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Many mathematical functions can be expressed as power series. Of particular interest in deriving Euler's Identity are the following:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots$$

Then, we have:

$$\mathrm{i} \cdot \sin(x) = i \cdot \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = ix - \frac{i \cdot x^3}{3!} + \frac{i \cdot x^5}{5!} - \frac{i \cdot x^7}{7!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = 1 + ix - \frac{x^2}{2!} - \frac{i \cdot x^3}{3!} + \frac{x^4}{4!} + \frac{i \cdot x^5}{5!} - \frac{x^6}{6!} - \frac{i \cdot x^7}{7!} + \cdots$$

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

This implies:

$$\boxed{\boldsymbol{e^{ix} = \cos x + i \sin x}} \quad \xrightarrow{\text{and, substituting } x = \pi \text{ yields:}} \quad \boxed{\boldsymbol{e^{i\pi} + 1 = 0}}$$

Logarithms of Negative Real Numbers and Complex Numbers

Natural Logarithm of a Negative Real Number

From Euler's Formula, we have:

$$e^{i\pi} = -1$$

Taking the natural logarithm of both sides gives:

$$\ln e^{i\pi} = \ln(-1) \qquad \text{which implies that} \qquad i\pi = \ln(-1)$$

Next, let x be a positive real number. Then:

$$\ln(-x) = \ln(-1 \cdot x) = \ln(-1) + \ln x$$

$$\boldsymbol{\ln(-x) = i\pi + \ln x}$$

Logarithm (Any Base) of a Negative Real Number

To calculate $\log_b(-x)$, use the change of base formula: $\log_b(m) = \frac{\log_a m}{\log_a b}$.

Let the new base be e to get: $\log_b(-x) = \dfrac{\ln(-x)}{\ln b}$

$$\boldsymbol{\log_b(-x) = \frac{i\pi + \ln x}{\ln b}}$$

Logarithm of a Complex Number (Principal Value)

Define $z = x + iy$ in polar form as: $z = re^{i\theta}$, where $r = \sqrt{x^2 + y^2}$ is the modulus (i.e., magnitude) of z and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ is the argument (i.e., angle), in radians, of complex number z. Then,

$$\boldsymbol{\ln z = \ln\left(re^{i\theta}\right) = \ln r + i\theta} \qquad \text{and} \qquad \boldsymbol{\log_b z = \frac{\ln r + i\theta}{\ln b}} \qquad \text{where, } \{-\pi < \theta \leq \pi\}$$

What Is i^i (i to the power of i)

Start with: $\boldsymbol{e^{i\pi} + 1 = 0}$ (Euler's Formula – special case)

Then: $e^{i\pi} = -1$

$$\sqrt{e^{i\pi}} = \sqrt{-1}$$

$$\left(e^{i\pi}\right)^{1/2} = i$$

$$e^{i\pi/2} = i$$

$$\left(e^{i\pi/2}\right)^{i} = i^i$$

$$e^{i^2\pi/2} = i^i$$

$$e^{-\pi/2} = i^i$$

Calculate $e^{-\pi/2}$ to obtain:

$$\boxed{\boldsymbol{i^i = e^{-\pi/2} \sim 0.20788 \sim \frac{1}{5}}}$$

So we see that it is possible to take an imaginary number to an imaginary power and return to the realm of real numbers.

Derivative of *e* to a Complex Power (e^z)

Start with: $z = x + iy$

$e^{iy} = \cos y + i \sin y$

Then: $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$

Cauchy-Riemann Equations

A complex function, $f(z) = u(x,y) + i \cdot v(x,y)$, is differentiable at point $z = z_0$ if and only if the functions u and v are differentiable and:

$$\frac{\partial u(z_0)}{\partial x} = \frac{\partial v(z_0)}{\partial y} \quad \text{and} \quad \frac{\partial u(z_0)}{\partial y} = -\frac{\partial v(z_0)}{\partial x}$$

These are called the Cauchy-Riemann Equations for the functions u and v:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{in Cartesian form}$$

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \qquad \text{in Polar form}$$

Derivative of e^z

For a differentiable complex function, $f(z) = u(x,y) + i \cdot v(x,y)$:

$$\frac{df}{dz} = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \left(\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\right)$$

Then, let $f(z) = e^z = e^x(\cos y + i \sin y)$:

$$u = e^x \cdot \cos y \quad \text{and} \quad v = e^x \cdot \sin y$$

$$\frac{d}{dz}(e^z) = \frac{\partial}{\partial x}(e^x \cdot \cos y) + i\frac{\partial}{\partial x}(e^x \cdot \sin y) = e^x \cdot \cos y + i \cdot e^x \cdot \sin y = e^z$$

So, $\frac{d}{dz}(e^z) = e^z$. Cool, huh?

Derivatives of a Circle

The general equation of a circle centered at the Origin is: $x^2 + y^2 = r^2$, where r is the radius of the circle.

First Derivative

$$x^2 + y^2 = r^2$$

Note that r^2 is a constant, so its derivative is zero. Using Implicit Differentiation (with respect to x), we get:

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

$$\boldsymbol{\frac{dy}{dx} = -\frac{x}{y}}$$

Second Derivative

We have a couple of options at this point. We could do implicit differentiation on $2x + 2y \cdot \frac{dy}{dx} = 0$, but given the simplicity of $\frac{dy}{dx} = -\frac{x}{y}$, let's work from there.

$$\frac{dy}{dx} = -\frac{x}{y}$$

Use the Quotient Rule, simplify and substitute in $\frac{dy}{dx} = -\frac{x}{y}$ in the expression.

$$\frac{d^2y}{dx^2} = -\left[\frac{y \cdot \frac{d}{dx}(x) - x \cdot \frac{dy}{dx}}{y^2}\right] = -\left[\frac{y - x\left(-\frac{x}{y}\right)}{y^2}\right] = -\left[\frac{\frac{y^2}{y} + \frac{x^2}{y}}{y^2}\right] = -\left[\frac{x^2 + y^2}{y^3}\right]$$

Notice that the numerator is equal to the left hand side of the equation of the circle. We can simplify the expression for the second derivative by substituting r^2 for $x^2 + y^2$ to get:

$$\boldsymbol{\frac{d^2y}{dx^2} = -\frac{r^2}{y^3}}$$

Derivatives of an Ellipse

The general equation of an ellipse centered at the Origin is: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a is the radius of the ellipse in the x-direction and b is the radius of the ellipse in the y-direction.

First Derivative

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which can also be written $b^2x^2+a^2y^2=a^2b^2$

Note that a^2b^2 is a constant, so its derivative is zero. Using Implicit Differentiation (with respect to x), we get:

$$2b^2x+2a^2y\cdot\frac{dy}{dx}=0$$

$$\boldsymbol{\frac{dy}{dx}=-\frac{b^2x}{a^2y}}$$

Second Derivative

Given the simplicity of $\frac{dy}{dx}=-\frac{b^2x}{a^2y}$, let's work from there to calculate $\frac{d^2y}{dx^2}$.

$$\frac{dy}{dx}=-\frac{b^2x}{a^2y}=-\frac{b^2}{a^2}\left(\frac{x}{y}\right)$$

Use the Quotient Rule, simplify and substitute in $\frac{dy}{dx}=-\frac{b^2x}{a^2y}$ in the expression.

$$\frac{d^2y}{dx^2}=-\frac{b^2}{a^2}\left[\frac{y\cdot\frac{d}{dx}(x)-x\cdot\frac{dy}{dx}}{y^2}\right]=-\frac{b^2}{a^2}\left[\frac{y-x\left(-\frac{b^2x}{a^2y}\right)}{y^2}\right]=-\frac{b^2}{a^2}\left[\frac{\frac{a^2y^2}{a^2y}+\frac{b^2x^2}{a^2y}}{y^2}\right]$$
$$=-\frac{b^2}{a^2}\left[\frac{a^2y^2+b^2x^2}{a^2y^3}\right]$$

Notice that the numerator inside the brackets is equal to the left hand side of the equation of the ellipse. We can simplify this expression by substituting a^2b^2 for $a^2y^2+b^2x^2$ to get:

$$\boldsymbol{\frac{d^2y}{dx^2}=-\frac{b^4}{a^2y^3}}$$

Derivatives of a Hyperbola

The general equation of a hyperbola with a vertical transverse axis, centered at the Origin is:
$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, where $(\pm a, 0)$ are the vertices of the hyperbola.

First Derivative

$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ which can also be written $b^2y^2 - a^2x^2 = a^2b^2$

Note that a^2b^2 is a constant, so its derivative is zero. Using Implicit Differentiation (with respect to x), we get:

$$2b^2y \cdot \frac{dy}{dx} - 2a^2x = 0$$

$$\boldsymbol{\frac{dy}{dx} = +\frac{a^2x}{b^2y}}$$

Second Derivative

Given the simplicity of $\frac{dy}{dx} = -\frac{x}{y}$, let's work from there to calculate $\frac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = \frac{a^2x}{b^2y} = \frac{a^2}{b^2}\left(\frac{x}{y}\right)$$

Use the Quotient Rule, simplify and substitute in $\frac{dy}{dx} = \frac{a^2x}{b^2y}$ in the expression.

$$\frac{d^2y}{dx^2} = \frac{a^2}{b^2}\left[\frac{y \cdot \frac{d}{dx}(x) - x \cdot \frac{dy}{dx}}{y^2}\right] = \frac{a^2}{b^2}\left[\frac{y - x\left(\frac{a^2x}{b^2y}\right)}{y^2}\right] = \frac{a^2}{b^2}\left[\frac{\frac{b^2y^2}{b^2y} - \frac{a^2x^2}{b^2y}}{y^2}\right]$$

$$= \frac{a^2}{b^2}\left[\frac{b^2y^2 - a^2x^2}{b^2y^3}\right]$$

Notice that the numerator inside the brackets is equal to the left hand side of the equation of the hyperbola. We can simplify this expression by substituting a^2b^2 for $b^2y^2 - a^2x^2$ to get:

$$\boldsymbol{\frac{d^2y}{dx^2} = \frac{a^4}{b^2y^3}}$$

Derivative of: $(x+y)^3 = x^3 + y^3$

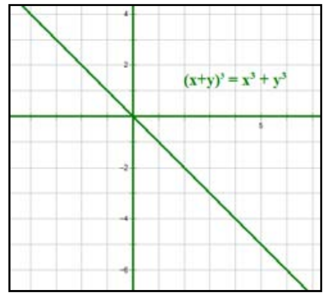
Starting expression: $(x+y)^3 = x^3 + y^3$

Expand the cubic of the binomial: $x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + y^3$

Subtract $(x^3 + y^3)$ from both sides: $3x^2y + 3xy^2 = 0$

Divide both sides by 3: $x^2y + xy^2 = 0$

Investigate this expression:



Factor it: $xy(x+y) = 0$

Solutions are the three lines: $x = 0,\ \ y = 0,\ \ y = -x$

Note the slopes of these lines: $\text{undefined},\ 0,\ -1$

Obtain the derivative:

Start with: $x^2y + xy^2 = 0$

Implicit differentiation: $\left(x^2 \cdot \frac{dy}{dx} + 2xy\right) + \left(x \cdot 2y\frac{dy}{dx} + y^2\right) = 0$

Rearrange terms: $(x^2 + 2xy)\frac{dy}{dx} + (2xy + y^2) = 0$

Solve for $\frac{dy}{dx}$: $\frac{dy}{dx} = \frac{-(2xy+y^2)}{x^2+2xy}$

Factored form: $\frac{dy}{dx} = \frac{-y(2x+y)}{x(x+2y)}$

Consider each solution separately:

$x = 0$: $\frac{dy}{dx} = \frac{-y(2\cdot 0+y)}{0(0+2y)} = \text{undefined}$

$y = 0$: $\frac{dy}{dx} = \frac{-0(2x+0)}{x(x+2\cdot 0)} = 0$

$y = -x$: $\frac{dy}{dx} = \frac{x(2x-x)}{x(x-2x)} = -1$

Conclusion:

$\frac{dy}{dx} = \frac{-y(2x+y)}{x(x+2y)}$ is an elegant way to describe the derivative of y with respect to x for the expression $(x+y)^3 = x^3 + y^3$ (which is not a function). However, it is noteworthy, that this derivative can only take on three possible values (if we allow "undefined" to count as a value) – undefined, 0 and -1.

Inflection Points of the PDF of the Normal Distribution

The equation for the Probability Density Function (PDF) of the Normal Distribution is:

$$\boldsymbol{P(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}}$$

where $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are the mean and standard deviation of the distribution.

$$P'(x) = \left(\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right) \cdot \frac{d}{dx}\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$P'(x) = P(x) \cdot \left(-\frac{2(x-\mu)}{2\sigma^2}\right)$$

$$\boldsymbol{P'(x) = -\frac{1}{\sigma^2} \cdot [P(x) \cdot (x-\mu)]}$$

$$P''(x) = -\frac{1}{\sigma^2}\left[P(x) \cdot \frac{d}{dx}(x-\mu) + (x-\mu) \cdot \frac{d}{dx}(P(x))\right]$$

$$P''(x) = -\frac{1}{\sigma^2}\left[P(x) \qquad + (x-\mu) \cdot \left(-\frac{1}{\sigma^2} \cdot P(x) \cdot (x-\mu)\right)\right]$$

$$\boldsymbol{P''(x)} = -\frac{1}{\sigma^2}\left[P(x) - P(x) \cdot \frac{(x-\mu)^2}{\sigma^2}\right] = \boldsymbol{-\frac{P(x)}{\sigma^2}\left[1 - \frac{(x-\mu)^2}{\sigma^2}\right]}$$

Setting $P''(x) = 0$, and noting that $P(x) \neq 0$ for all values of x, we get:

$1 - \frac{(x-\mu)^2}{\sigma^2} = 0$ So that: $\boldsymbol{x = \mu \pm \sigma.}$

Further, noting that the value of the second derivative changes signs at each of these values, we conclude that inflection points exist at $\boldsymbol{x = \mu \pm \sigma.}$

In English, the inflection points of the Probability Density Function of the Normal Distribution exist at points one standard deviation above or below the mean.

Appendix A
Key Definitions in Calculus

Absolute Maximum

See entry on *Global Maximum.* May also simply be called the *"maximum."*

Absolute Minimum

See entry on *Global Minimum.* May also simply be called the *"minimum."*

Antiderivative

Also called the *indefinite integral* of a function, $f(x)$, an *antiderivative* of $f(x)$ is a function $F(x)$, such that $F'(x) = f(x)$ on an interval of x.

The *general antiderivative* of $f(x)$ is the antiderivative expressed as a function which includes the addition of a constant C, which is called the *constant of integration*.

Example: $F(x) = 2x^3$ is an antiderivative of $f(x) = 6x^2$ because $F'(x) = f(x)$.

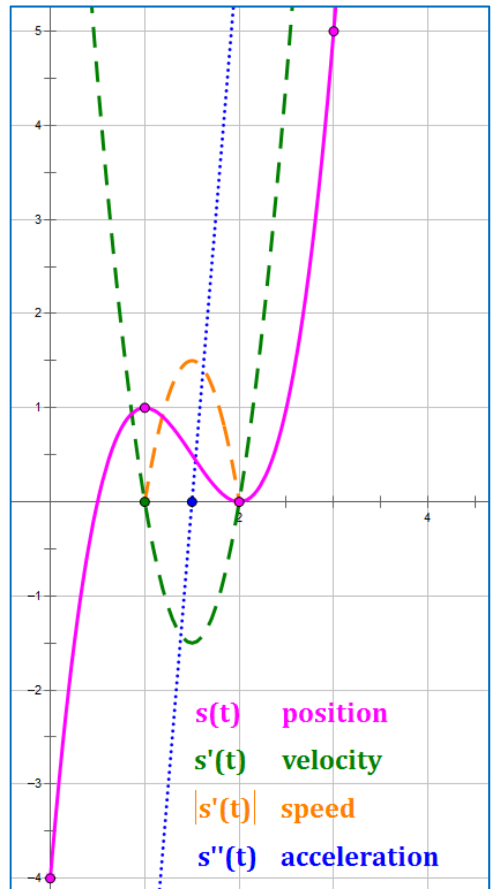
$F(x) = 2x^3 + C$ is the general antiderivative of $f(x) = 6x^2$ because $F'(x) = f(x)$ for all values of C.

Notation: the antiderivative of a function, $f(x)$, is expressed as: $\boldsymbol{F(x) = \int f(x)\, dx}$.

Ault Table

Named for A'Laina Ault, the Math Department Chair at Damonte Ranch High School in Reno, Nevada, an *Ault Table* is a chart that shows the signs and the behavior of a function and its derivatives over key intervals of the independent variable (usually x or t). It is very useful in curve sketching because it makes the process of finding extrema and inflection points relatively easy. The steps to building an Ault Table are:

1. Calculate the first and second *derivatives* of the function being considered. Additional derivatives may be taken if needed.
2. Find the *zeros* of each derivative; these form the interval endpoints for the table. Note that the zeros of the first derivative are *critical values*, representing potential *maxima* and *minima*, and the zeros of the second derivative are potential *inflection points*.
3. Arrange the zeros of the first two derivatives in numerical order, and create mutually exclusive open intervals with the zeros as endpoints. If appropriate, include intervals extending to $-\infty$ and/or ∞.
4. Create a set of rows as shown in the table on the next page. At this point the boxes in the table will be empty.
5. Determine the sign of each derivative in each interval and record that information in the appropriate box using a "+" or a "−".
6. Use the signs determined in Step 5 to identify for each interval a) whether the function is increasing or decreasing (green lines in the table), b) whether the first derivative is increasing or decreasing (red lines in the table), c) whether the function is concave up or down (bottom red line in the table), and d) the shape of the curve on the interval.



An Ault table facilitates the graphing of a function like the one above:

$s(t) = 2t^3 - 9t^2 + 12t - 4$

From the information in the table, you can determine the location of all extrema and inflection points of the curve. You can also determine where the speed is positive; the signs of both the first and second derivatives are the same.

An example is provided on the next page:

Example: develop an Ault Table for the function: $s(t) = 2t^3 - 9t^2 + 12t - 4$

First find the key functions:

$s(t) = 2t^3 - 9t^2 + 12t - 4$ Position function

$s'(t) = 6t^2 - 18t + 12$ Velocity function

$|s'(t)| = |6t^2 - 18t + 12|$ Speed function

$s''(t) = 12t - 18$ Acceleration function

Next, find the function's critical values, inflection points, and maybe a couple more points.

$s(t) = 2t^3 - 9t^2 + 12t - 4$ $s(0) = -4$

$s'(t) = 6(t-1)(t-2)$ $s'(t) = 0 \;\Rightarrow$ Critical Values of t are: $t = \{1, 2\}$

Critical Points are: $\{(1,1), (2,0)\}$

$s''(t) = 6(2t-3)$ $s''(t) = 0 \;\Rightarrow$ Inflection Point at: $t = 1.5$

$s(t) = 2t^3 - 9t^2 + 12t - 4$ $s(3) = 5$, just to get another point to plot

Then, build an Ault Table with intervals separated by the key values:

Key values of t that define the intervals in the table are $t = \{1,\ 1.5,\ 2\}$

Note: Identify the signs (i.e., "+, "−") first. The word descriptors are based on the signs.

$\boldsymbol{s(t) = 2t^3 - 9t^2 + 12t - 4}$				
	$(\mathbf{0}, \mathbf{1})$	$(\mathbf{1}, \mathbf{1.5})$	$(\mathbf{1.5}, \mathbf{2})$	$(\mathbf{2}, \infty)$
$\boldsymbol{s(t)}$	**increasing**	**decreasing**	**decreasing**	**increasing**
$\boldsymbol{s'(t)}$ **and is:**	$+$ **decreasing**	$-$ **decreasing**	$-$ **increasing**	$+$ **increasing**
$\boldsymbol{s''(t)}$ **so $\boldsymbol{s(t)}$ is:**	$-$ **concave down**	$-$ **concave down**	$+$ **concave up**	$+$ **concave up**
Curve Shape				

Results. This function has:

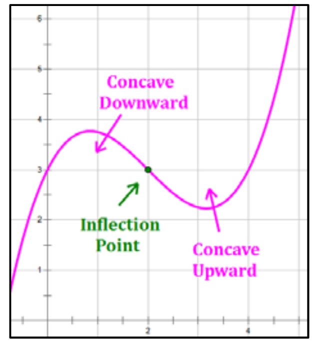
- A maximum at $t = 1$.
- A minimum at $t = 2$.
- An inflection point at $t = 1.5$.

Concavity

A function, f, is *concave upward* on an interval if $f'(x)$ is increasing on the interval, i.e., if $f''(x) > 0$.

A function, f, is *concave downward* on an interval if $f'(c)$ is decreasing on the interval, i.e., if $f''(x) < 0$.

Concavity changes at inflection points, from upward to downward or from downward to upward.



Continuity

A function, f, is *continuous* at $x = c$ iff:

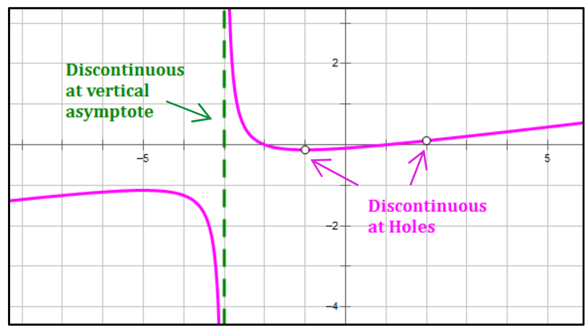
a. $f(c)$ is defined,

b. $\lim_{x \to c} f(x)$ exists, and

c. $\lim_{x \to c} f(x) = f(c)$

d. If $x = a$ is an endpoint, then the limit need only exist from the left or the right.

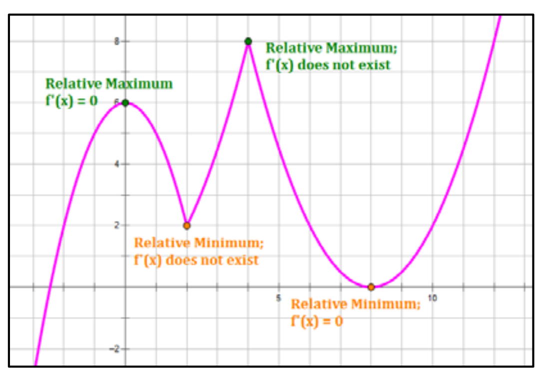
Basically, the function value and limit at a point must both exist and be equal to each other.



The curve shown is continuous everywhere except at the holes and the vertical asymptote.

Critical Numbers or Critical Values (and Critical Points)

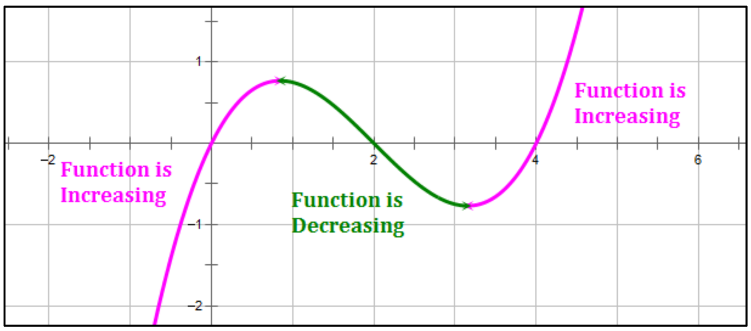
If a function, f, is defined at c, then the *critical numbers* (also called *critical values*) of f are x-values where $f'(c) = 0$ and where $f'(c)$ does not exist (i.e., f is not differentiable at c). This includes x-values where the slope of the curve is horizontal, and where cusps and discontinuities exist in an interval.



The points where the critical numbers exist are called *critical points*. *Note: endpoints are excluded from this definition, but must also be tested in cases where the student seeks an absolute (i.e., global) maximum or minimum of an interval.*

Decreasing Function

A function, f, is *decreasing* on an interval if for any two values in the interval, a and b, with $a < b$, it is true that $f(a) > f(b)$.



Degree of a Differential Equation

The *degree of a differential equation* is the power of the highest derivative term in the equation. Contrast this with the *order of a differential equation*.

Examples:

- $\frac{d^2y}{dx^2} + 4\left(\frac{dy}{dx}\right) + 4y = 0$ Degree $= 1$
- $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^3 = y + ax(x^2 + y^2)$ Degree $= 2$
- $3\left(\frac{d^4y}{dx^4}\right)^5 + \left(\frac{d^3y}{dx^3}\right)^2 - 2y = 2\cos x$ Degree $= 5$

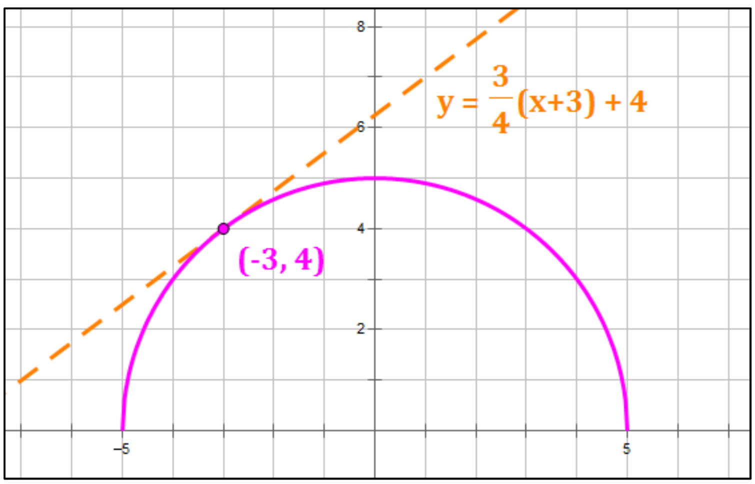
Derivative

The measure of the slope of a curve at each point along the curve. The derivative of a function $f(x)$ is itself a function, generally denoted $f'(x)$ or $\frac{dy}{dx}$. The *derivative* provides the instantaneous rate of change of a function at the point at which it is measured.

The derivative function is given by either of the two following limits, which are equivalent:

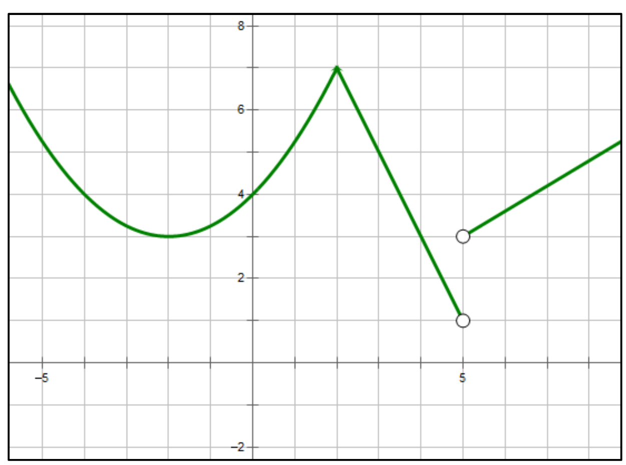
$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{or} \qquad \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

In the figure below, the derivative of the curve $f(x) = \sqrt{25 - x^2}$ at $(-3, 4)$ is the slope of the tangent line at $(-3, 4)$, which is $\frac{3}{4}$.



Differentiable

A function is differentiable at a point, if a derivative can be taken at that point. A function is <u>not</u> differentiable at any x-value that is not in its domain, at discontinuities, at sharp turns and where the curve is vertical.

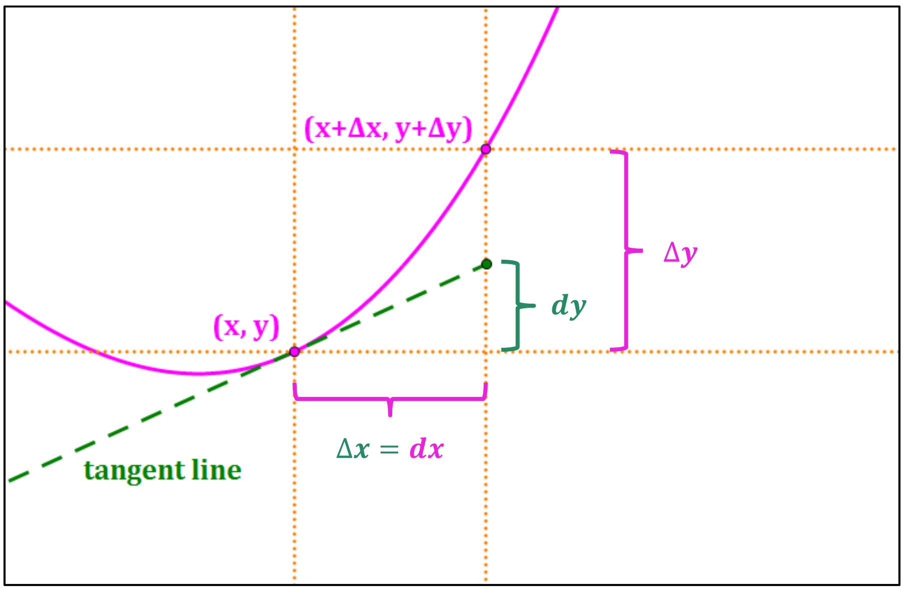


To find where a function is not differentiable by inspection, look for points of discontinuity, sharp turns, and vertical slopes in the curve. In the curve shown at right, the curve is not differentiable at the points of discontinuity $(x = 5)$ nor at the cusp $(x = 2)$.

Differential

Consider a function $f(x)$, that is differentiable on an open interval around x. Δx and Δy represent small changes in the variables x and y around x on f. Then,

- The *differential* of x is denoted as dx, and $dx = \Delta x$.
- The *differential* of y is denoted as dy, and $dy = f'(x) \cdot dx$
- Δy is the actual change is y resulting from a change in x of Δx. dy is an approximation of Δy.



Differential Equation

An equation which includes variables and one or more of their derivatives.

An *ordinary differential equation (ODE)* is a differential equation that includes an independent variable (e.g., x), a dependent variable (e. g., y), and one or more derivatives of the dependent varaiable, (e.g., $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$, etc.).

If the differential equation includes partial derivatives, it is a *partial differential equation (PDE)*, and not an ordinary differential equation. See Chapter 10 for more definitions.

Examples:

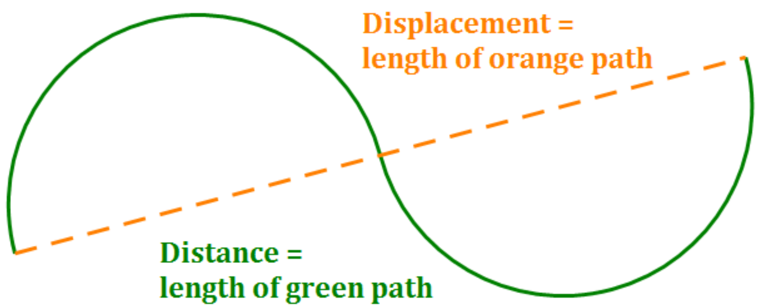
- $\frac{dy}{dx} = e^x$
- $\frac{dy}{dx} = y + ax(x^2 + y^2)$
- $\frac{1}{4}\left(\frac{d^2y}{dx^2}\right) + x\frac{dy}{dx} + 1 = 0$
- $3\left(\frac{d^2y}{dx^2}\right) + \frac{dy}{dx} - 2y = 2\cos x$

Displacement

Displacement is a measure of the shortest path between two points. So if you start at Point A and end at Point B, the length of the line segment connecting them is the displacement.

To get displacement from velocity:

- Integrate velocity over the entire interval, without any breaks.



Distance

Distance is a measure of the length of the path taken to get from one point to another. So, traveling backward adds to distance and reduces displacement.

To get distance from velocity, over an interval $[a, b]$:

- Integrate velocity over the $[a, b]$ in pieces, breaking it up at each point where velocity changes sign from "$+$" to "$-$" or from "$-$" to "$+$".
- Take the absolute value of each separate definite integral to get the distance for that interval.
- Add the distances over each interval to get the total distance.

e

e is the base of the *natural logarithms*. It is a *transcendental number*, meaning that it is not the root of any polynomial with integer coefficients.

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \qquad e = \lim_{n\to\infty}\left(\frac{n}{\sqrt[n]{n!}}\right) \qquad \frac{d}{dx}(e^x) = e^x \qquad \int_1^e \frac{1}{x}dx = 1$$

$$e = \sum_{k=0}^{\infty}\frac{1}{k!} = 1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+\cdots$$

$$e = \left[\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\right]^{-1} = \frac{1}{1-1+\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}+\cdots}$$

Euler's Equation:

$e^{i\pi}+1=0$ shows the interconnection of five seemingly unrelated mathematical constants.

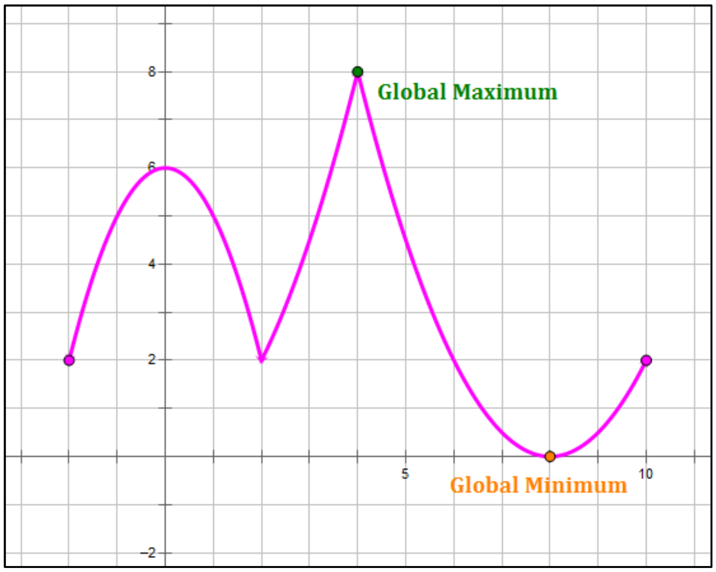
Decimal Expansion of e:

e =2.7 1828 1828 4590 4523 5360 2874 7135 2662 4977 5724 7093 6999 5957 4966 …

The web site http://antwrp.gsfc.nasa.gov/htmltest/gifcity/e.2mil shows the decimal expansion of e to over 2 million digits.

Global Maximum

A *global maximum* is the function value at point c on an interval if $f(x) < f(c)$ for all x in the interval. That is, $f(c)$ is a global maximum if there is an interval containing c where $f(c)$ is the greatest value in the interval. Note that the interval may contain multiple relative maxima but only one global maximum.



Global Minimum

A *global minimum* is the function value at point c on an interval if $f(x) > f(c)$ for all x in the interval. That is, $f(c)$ is a global minimum if there is an interval containing c where $f(c)$ is the least value in the interval. Note that the interval may contain multiple relative minima but only one global minimum.

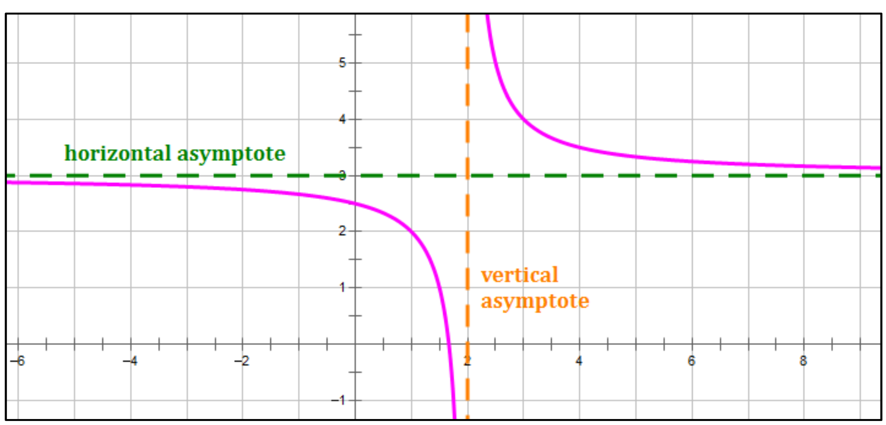
Horizontal Asymptote

If:

$$\lim_{x\to\infty} f(x) = L, \text{ or}$$

$$\lim_{x\to-\infty} f(x) = L,$$

then the line $y = L$ is a *horizontal asymptote* of f.



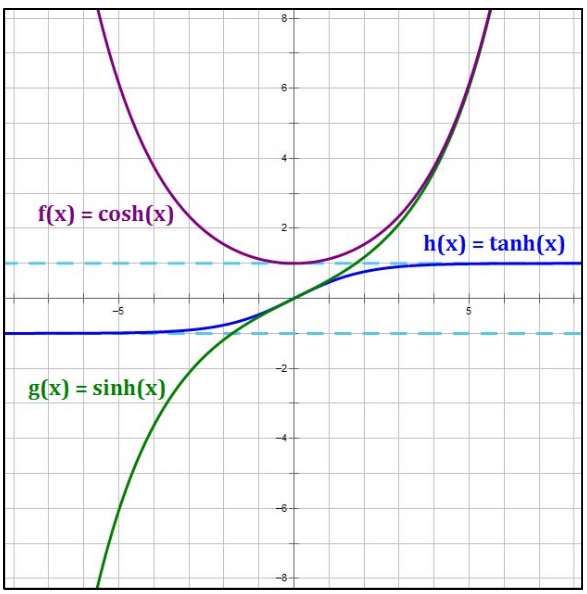
Hyperbolic Functions

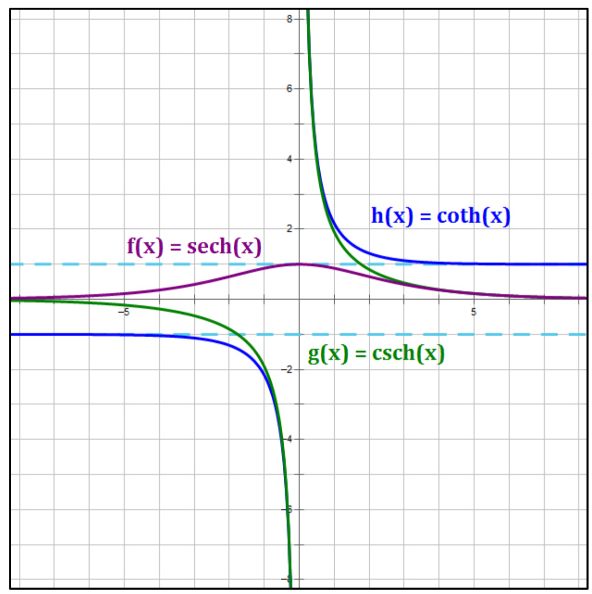
The set of *hyperbolic functions* relate to the unit hyperbola in much the same way that trigonometric functions relate to the unit circle. Hyperbolic functions have the same shorthand names as their corresponding trigonometric functions, but with an "h" at the end of the name to indicate that the function is hyperbolic. The names are read "hyperbolic sine," "hyperbolic cosine," etc.

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} \qquad \operatorname{csch} x = \frac{2}{e^x - e^{-x}}$$

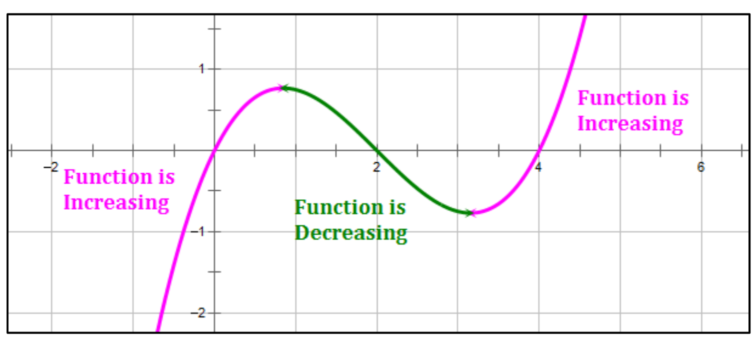
Graphs of Hyperbolic Functions





Increasing Function

A function, f, is *increasing* on an interval if for any two values in the interval, a and b, with $a < b$, it is true that $f(a) < f(b)$.

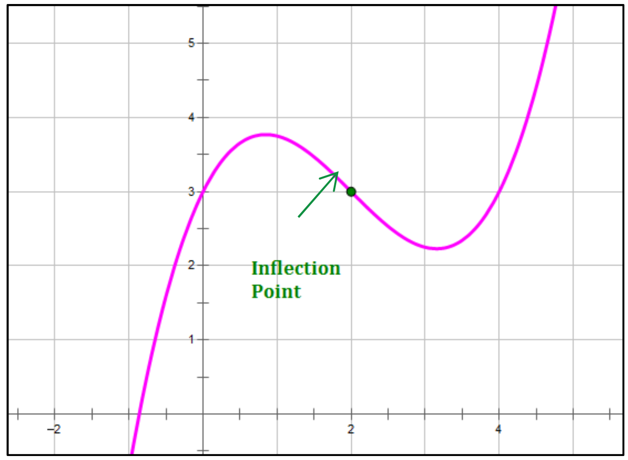


Inflection Point

An *inflection point* is a location on a curve where concavity changes from upward to downward or from downward to upward.

At an inflection point, the curve has a tangent line and $f''(x) = 0$ or $f''(x)$ does not exist.

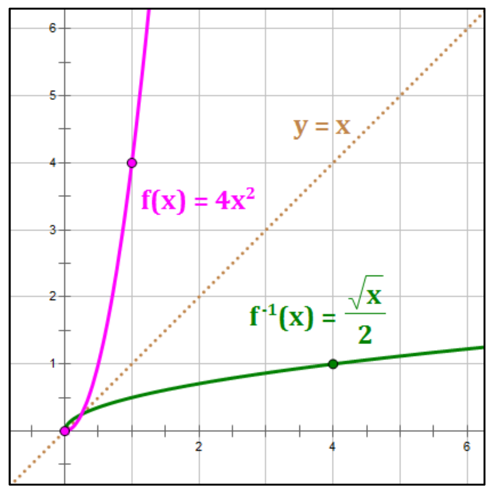
However, it is not necessarily true that if $f''(x) = 0$, then there is an inflection point at $x = c$.



Inverse Function

Two functions $f(x)$ and $g(x)$ are *inverses* if and only if:

- $f(g(x)) = x$ for every x in the domain of g, and
- $g(f(x)) = x$ for every x in the domain of f.



Important points about inverse functions:

- Each function is a reflection of the other over the line $y = x$.
- The domain of each function is the range of the other. Sometimes a domain restriction is needed to make this happen.
- If $f(a) = b$, then $f^{-1}(b) = a$.
- The slopes of inverse functions at a given value of x are reciprocals.

Local Maximum

See entry on *Relative Maximum.*

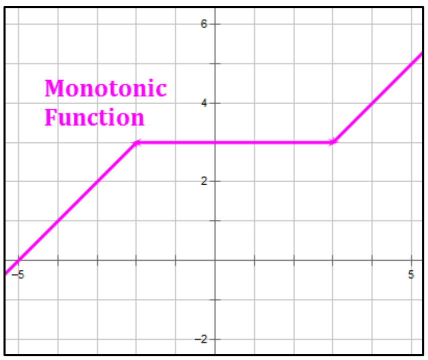
Local Minimum

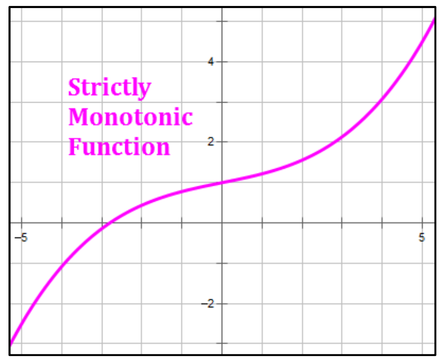
See entry on *Relative Minimum.*

Monotonic Function

A function f is *monotonic* if it is either entirely non-increasing or entirely non-decreasing. The derivative of a monotonic function never changes sign.

A *strictly monotonic* function is either entirely increasing or entirely decreasing. The derivative of a strictly monotonic function is either always positive or always negative. Strictly monotonic functions are also one-to-one.



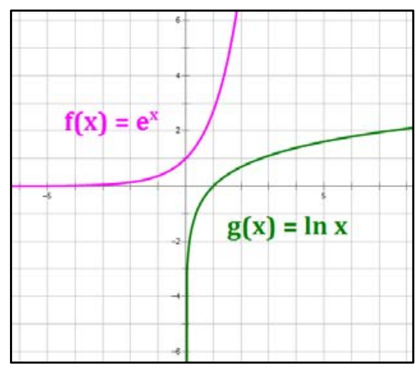


Natural Exponential Function

The *natural exponential function* is defined as:

$$f(x) = e^x.$$

It is the inverse of the natural logarithmic function.



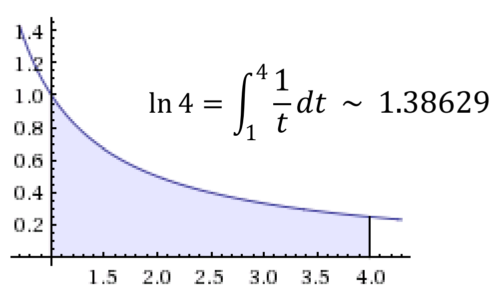
Natural Logarithmic Function

The *natural logarithmic function* is defined as:

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0.$$

The base of the natural logarithm is e. So,

$$\ln x = \log_e x$$



One-to-One Function

A function f is *one-to-one* if:

- for every x in the domain of f, there is exactly one y such that $f(x) = y$, and
- for every y in the range of f, there is exactly one x such that $f(x) = y$.

A function has an inverse if and only if it is one-to-one. One-to-one functions are also monotonic. Monotonic functions are not necessarily one-to-one, but strictly monotonic functions are necessarily one-to-one.

Order of a Differential Equation

The *order of a differential equation* is the highest derivative that occurs in the equation. Contrast this with the *degree of a differential equation*.

Examples:

- $\frac{d^4y}{dx^4} + 4\left(\frac{d^2y}{dx^2}\right) + 4y = 0$ Order $= 4$
- $\frac{dy}{dx} = y + ax(x^2 + y^2)$ Order $= 1$
- $3\left(\frac{d^2y}{dx^2}\right) + \frac{dy}{dx} - 2y = 2\cos x$ Order $= 2$

Ordinary Differential Equation (ODE)

An *ordinary differential equation* is one that involves a single independent variable.

Examples of ODEs:

- $\frac{d^4y}{dx^4} + 4\left(\frac{d^2y}{dx^2}\right) + 4y = 0$
- $\frac{dy}{dx} = y + ax(x^2 + y^2)$
- $3\left(\frac{d^2y}{dx^2}\right) + \frac{dy}{dx} - 2y = 2\cos x$

Not ODEs (Partial Differential Equations):

- $\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} - z$
- $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$
- $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

Partial Differential Equation (PDE)

A *partial differential equation* is one that involves more than one independent variable.

Examples of PDEs:

- $\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} - z$
- $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$
- $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

Position Function

A *position function* is a function that provides the location (i.e., position) of a point moving in a straight line, in a plane or in space. The position function is often denoted $s(t)$, where t is time, the independent variable. When position is identified along a straight line, we have:

$s(t)$	Position function
$s'(t)$	Velocity function (rate of change in position; may be positive, negative, or zero)
$\lvert s'(t)\rvert$	Speed function (absolute value of velocity; it is zero or positive by definition)
$s''(t)$	Acceleration function (rate of change in velocity)
$s'''(t)$	Jerk function (rate of change in acceleration)

Note that the inverse relationships hold for the functions as well. For example, consider the position function $s(t)$ and the velocity funtion $v(t)$:

$$v(t) = s'(t) \qquad \text{and} \qquad s(t) = \int v(t)\,dt$$

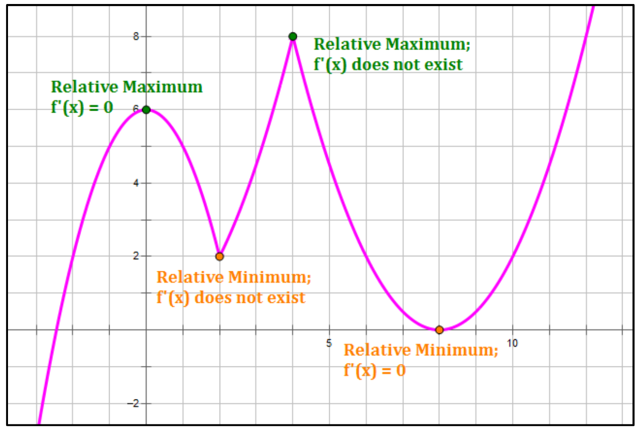
General Case of Integrating the Position Function in Problems Involving Gravity

Given intial position $s(0)$, and intial velocity $v(0)$, the position function is given as:

$s(t) = -16t^2 + v(0)t + s(0)$ where all functions involve the units feet and seconds.

Note: The force of gravity is $-32\ ft/sec^2$ or $-9.8\ m/sec^2$.

Relative Maximum



A *relative maximum* is the function value at point c in an open interval if $f(c-\delta) < f(c)$ and $f(c+\delta) < f(c)$ for arbitrarily small δ. That is, $f(c)$ is a relative maximum if there is an open interval containing c where $f(c)$ is the greatest value in the interval.

Relative Minimum

A *relative minimum* is the function value at point c in an open interval if $f(c-\delta) > f(c)$ and $f(c+\delta) > f(c)$ for arbitrarily small δ. That is, $f(c)$ is a relative minimum if there is an open interval containing c where $f(c)$ is the least value in the interval.

Riemann Integral

If $\sum_{i=1}^{n} f(x_i^*) \cdot \Delta x_i$ is a Riemann Sum (see the entry on "Riemann Sum" below), then the corresponding definite integral, $\int_a^b f(x)dx$ is called the *Riemann Integral* of $f(x)$ on the interval $[a,b]$. Riemann Integrals in one, two and three dimensions are:

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x_i$$

$$\iint f(x,y)\,dA = \lim_{\max \Delta A_i \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*) \cdot \Delta A_i$$

$$\iiint f(x,y,z)\,dV = \lim_{\max \Delta V_i \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*) \cdot \Delta V_i$$

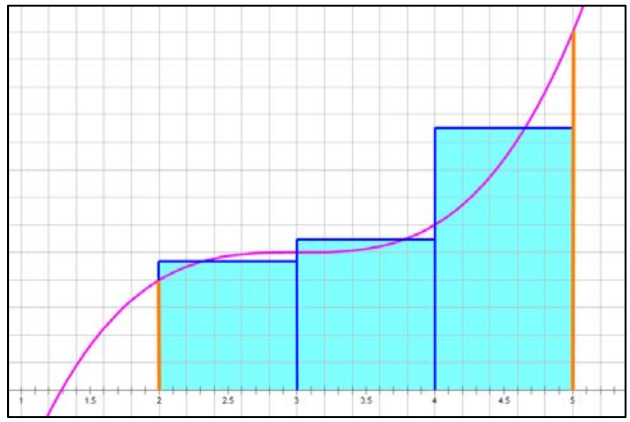
Riemann Sum

A *Riemann Sum* is the sum of the areas of a set of rectangles that can be used to approximate the area under a curve over a closed interval.

Consider a closed interval $[a, b]$ on x that is partitioned into n sub-intervals of lengths $\Delta x_1,\ \Delta x_2,\ \Delta x_3,\ \ldots\ \Delta x_n$. Let x_i^* be any value of x on the i-th sub-interval. Then, the Riemann Sum is given by:

$$S = \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x_i$$

A graphical representation of a Riemann sum on the interval $[2, 5]$ is provided at right.



Note that the *area under a curve* from $x = a$ to $x = b$ is:

$$\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x_i = \int_a^b f(x) dx$$

The largest Δx_i is called the *mesh size* of the partition. A typical Riemann Sum is developed with all Δx_i the same (i.e., constant mesh size), but this is not required. The resulting definite integral, $\int_a^b f(x) dx$ is called the *Riemann Integral* of $f(x)$ on the interval $[a, b]$.

Scalar Field

A *Scalar Field* in three dimensions provides a value at each point in space. For example, we can measure the temperature at each point within an object. The temperature can be expressed as T=φ(x,y,z). (note: φ is the Greek letter phi, corresponding to the English letter "f".)

Separation of Variables

Separation of Variables is a technique used to assist in the solution of differential equations. The process involves using algebra to collect all terms involving one variable on one side of an equation and all terms involving the other variable on the other side of an equation.

Example:

Original differential equation: $\dfrac{dy}{dx} = \dfrac{y}{2\sqrt{x}}$

Revised form with variables separated: $\dfrac{dy}{y} = \dfrac{dx}{2\sqrt{x}}$

Singularity

A *singularity* is a point at which a mathematical expression or other object is not defined or fails to be well-behaved. Typically, singularities exist at discontinuities.

Example:

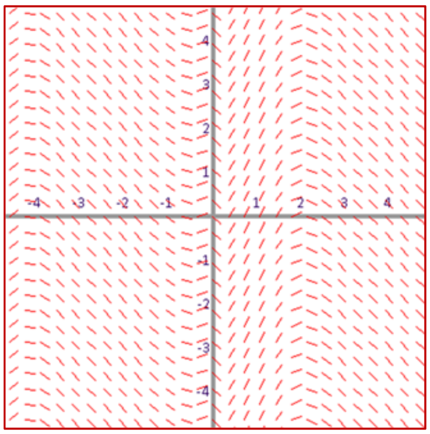
In evaluating the following integral, we notice that $e^{1/x}$ does not exist at $x = 0$. We say, then, that $e^{1/x}$ has a singularity at $x = 0$. Special techniques must often be employed to solve integrals with singularities.

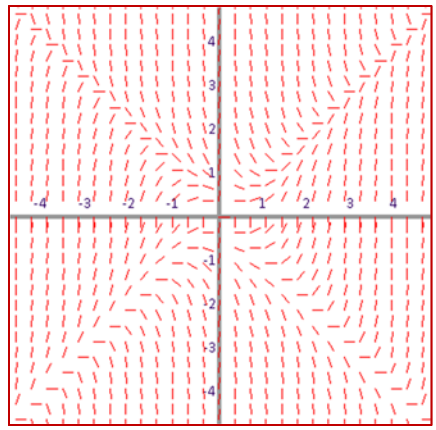
$$\int_{-1}^{1} e^{1/x}\, dx$$

Slope Field

A *slope field* (also called a *direction field*) is a graphical representation of the slopes of a curve at various points that are defined by a differential equation. Each position in the graph (i.e., each point (x, y)) is represented by a line segment indicating the slope of the curve at that point.

Examples: $\frac{dy}{dx} = e^{\sin x} \cos x + \sin x$ $\frac{dy}{dx} = x^2 - y^2$





If you know a point on a curve and if you have its corresponding slope field diagram, you can plot your point and then follow the slope lines to determine the curve. Slope field plotters are available online at:

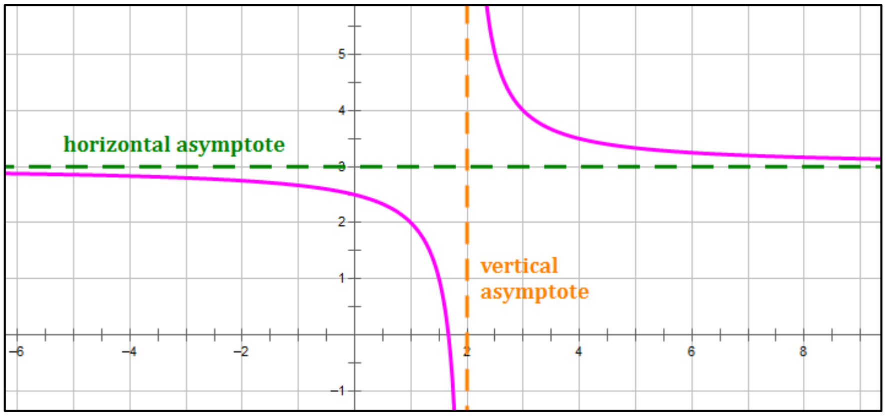
- http://www.mathscoop.com/calculus/differential-equations/slope-field-generator.php
- http://www.geogebratube.org/student/m42741

Vector Field

A *Vector Field* in three dimensions provides a vector at each point in space. For example, we can measure a magnetic field (magnitude and direction of the magnetic force) at each point in space around a charged particle. The magnetic field can be expressed as $\vec{M} = \vec{V}(x, y, z)$. Note that the half-arrow over the letters M and V indicate that the function generates a vector field.

Vertical Asymptote

If $\lim_{x \to c^-} f(x) = \pm\infty$ or $\lim_{x \to c^+} f(x) = \pm\infty$, then the line $x = c$ is a *vertical asymptote* of f.



Appendix B
Key Theorems in Calculus

Functions and Limits

Inverse Function Theorem

A function has an inverse function if and only if it is one-to-one.

Intermediate Value Theorem (IVT)

If

- a function, f, is continuous on the closed interval $[a, b]$, and
- d is a value between $f(a)$ and $f(b)$,

Then

- there is a value c in $[a, b]$ such that $f(c) = d$.

Extreme Value Theorem (EVT)

If

- a function, f, is continuous on the closed interval $[a, b]$,

Then

- f has both an absolute maximum and an absolute minimum on $[a, b]$.

Squeeze Theorem (Limits):

If

- $g(x) \leq f(x) \leq h(x)$, and
- $\lim_{x \to c} g(x) = L = \lim_{x \to c} h(x)$

Then

- $\lim_{x \to c} f(x) = L$

Differentiation

Rolle's Theorem

If

- a function, f, is continuous on the closed interval $[a, b]$, and
- f is differentiable on the open interval (a, b), and
- $f(a) = f(b)$,

Then

- there is at least one value c in (a, b) where $f'(c) = 0$.

Mean Value Theorem (MVT)

If

- a function, f, is continuous on the closed interval $[a, b]$, and
- f is differentiable on the open interval (a, b),

Then

- There is at least one value c in (a, b) where $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

Increasing and Decreasing Interval Theorem

If

- a function, f, is continuous on the closed interval $[a, b]$, and
- f is differentiable on the open interval (a, b),

Then

- If $f'(x) > 0$ for every $x \in (a, b)$, then f is increasing on $[a, b]$.
- If $f'(x) < 0$ for every $x \in (a, b)$, then f is decreasing on $[a, b]$.
- If $f'(x) = 0$ for every $x \in (a, b)$, then f is constant on $[a, b]$.

Concave Interval Theorem

If

- a function, f, is continuous on the closed interval $[a, b]$, and
- $f''(x)$ exists on the open interval (a, b),

Then

- If $f''(x) > 0$ for every $x \in (a, b)$, then f is concave upward on $[a, b]$.
- If $f''(x) < 0$ for every $x \in (a, b)$, then f is concave downward on $[a, b]$.

First Derivative Test (for finding extrema)

If

- a function, f, is continuous on the open interval (a, b), and
- c is a critical number $\in (a, b)$,
- f is differentiable on the open interval (a, b), except possibly at *c,*

Then

- If $f'(x)$ changes from positive to negative at c, then $f(c)$ is a relative maximum.
- If $f'(x)$ changes from negative to positive at c, then $f(c)$ is a relative minimum.

Second Derivative Test (for finding extrema)

If

- a function, f, is continuous on the open interval (a, b), and
- $c \in (a, b)$, and
- $f'(c) = 0$ and $f''(c)$ exists*,*

Then

- If $f''(c) < 0$, then $f(c)$ is a relative maximum.
- If $f''(c) > 0$, then $f(c)$ is a relative minimum.

Inflection Point Theorem

If

- a function, f, is continuous on the open interval (a, b), and
- $c \in (a, b)$, and
- $f''(c) = 0$ or $f''(c)$ does not exist,

Then

- $(c, f(c))$ may be an inflection point of f.

Inverse Function Continuity and Differentiability

If

- a function, f, has an inverse,

Then

- If f is continuous on its domain, then so is f^{-1} on its domain.
- If f is increasing on its domain, then so is f^{-1} on its domain.
- If f is decreasing on its domain, then so is f^{-1} on its domain.
- If f is differentiable on its domain, then so is f^{-1} on its domain (wherever $f'(x) \neq 0$).
  Note: this exception exists because the derivatives of f and f' are inverses.

Derivative of an Inverse Function

If

- a function, f, is differentiable at $x = a$, and
- f has an inverse function g, and
- $f(a) = b$,

Then

- $f'(a) = \dfrac{1}{g\prime(b)}$ (i.e., the derivatives of inverse functions are reciprocals).

Integration

First Fundamental Theorem of Calculus

If

- $f(x)$ is a continuous function on $[a, b]$,
- $F(x)$ is any antiderivative of $f(x)$, then

Then

- $\displaystyle\int_a^b f(x)dx = F(b) - F(a)$

Second Fundamental Theorem of Calculus

If

- $f(x)$ is a continuous function on $[a, b]$,

Then

- For every $x \in [a, b]$, $\displaystyle\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

Mean Value Theorem for Integrals (MVT)

If

- $f(x)$ is a continuous function on $[a, b]$,

Then

- there is a value $c \in [a, b]$, such that

$$\int_a^b f(x)\,dx = (b - a) \cdot f(c)$$

Appendix C

Summary of Key Derivatives and Integrals

Derivatives of Special Functions

Common Functions

Power Rule

$$\frac{d}{dx}(x^n) = n \cdot x^{n-1}$$

$$\frac{d}{dx}(u^n) = n \cdot u^{n-1}\frac{du}{dx}$$

Exponential and Logarithmic Functions $(a > 0, a \neq 1)$

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}e^u = e^u \cdot \frac{du}{dx}$$

$$\frac{d}{dx}a^x = a^x \cdot \ln a$$

$$\frac{d}{dx}a^u = a^u \cdot \ln a \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

$$\frac{d}{dx}\ln u = \frac{1}{u} \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\log_a x = \frac{1}{x\ \ln a}$$

$$\frac{d}{dx}\log_a u = \frac{1}{u\ \ln a} \cdot \frac{du}{dx}$$

Trigonometric Functions

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\sin u = \cos u \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\cos u = -\sin u \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\tan u = \sec^2 u \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\cot u = -\csc^2 u \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\sec x = \sec x\ \tan x$$

$$\frac{d}{dx}\sec u = \sec u\ \tan u \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\csc x = -\csc x \cot x$$

$$\frac{d}{dx}\csc u = -\csc u \cot u \cdot \frac{du}{dx}$$

Derivatives of Special Functions

Common Functions

Inverse Trigonometric Functions

$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$	**Angle in Q I or Q IV**
$\frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\cos^{-1} u = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$	**Angle in Q I or Q II**
$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$	$\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2} \cdot \frac{du}{dx}$	**Angle in Q I or Q IV**
$\frac{d}{dx}\cot^{-1} x = \frac{-1}{1+x^2}$	$\frac{d}{dx}\cot^{-1} u = \frac{-1}{1+u^2} \cdot \frac{du}{dx}$	**Angle in Q I or Q II**
$\frac{d}{dx}\sec^{-1} x = \frac{1}{\lvert x\rvert \sqrt{x^2-1}}$	$\frac{d}{dx}\sec^{-1} u = \frac{1}{\lvert u\rvert \sqrt{u^2-1}} \cdot \frac{du}{dx}$	**Angle in Q I or Q II**
$\frac{d}{dx}\csc^{-1} x = \frac{-1}{\lvert x\rvert \sqrt{x^2-1}}$	$\frac{d}{dx}\csc^{-1} u = \frac{-1}{\lvert u\rvert \sqrt{u^2-1}} \cdot \frac{du}{dx}$	**Angle in Q I or Q IV**

$\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{a^2-x^2}}$	$\frac{d}{dx}\sin^{-1}\left(\frac{u}{a}\right) = \frac{1}{\sqrt{a^2-u^2}} \cdot \frac{du}{dx}$	**Angle in Q I or Q IV**
$\frac{d}{dx}\cos^{-1}\left(\frac{x}{a}\right) = \frac{-1}{\sqrt{a^2-x^2}}$	$\frac{d}{dx}\cos^{-1}\left(\frac{u}{a}\right) = \frac{-1}{\sqrt{a^2-u^2}} \cdot \frac{du}{dx}$	**Angle in Q I or Q II**
$\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{a^2+x^2}$	$\frac{d}{dx}\tan^{-1}\left(\frac{u}{a}\right) = \frac{a}{a^2+u^2} \cdot \frac{du}{dx}$	**Angle in Q I or Q IV**
$\frac{d}{dx}\cot^{-1}\left(\frac{x}{a}\right) = \frac{-a}{a^2+x^2}$	$\frac{d}{dx}\cot^{-1}\left(\frac{u}{a}\right) = \frac{-a}{a^2+u^2} \cdot \frac{du}{dx}$	**Angle in Q I or Q II**
$\frac{d}{dx}\sec^{-1}\left(\frac{x}{a}\right) = \frac{a}{\lvert x\rvert \sqrt{x^2-a^2}}$	$\frac{d}{dx}\sec^{-1}\left(\frac{u}{a}\right) = \frac{a}{\lvert u\rvert \sqrt{u^2-a^2}} \cdot \frac{du}{dx}$	**Angle in Q I or Q II**
$\frac{d}{dx}\csc^{-1}\left(\frac{x}{a}\right) = \frac{-a}{\lvert x\rvert \sqrt{x^2-a^2}}$	$\frac{d}{dx}\csc^{-1}\left(\frac{u}{a}\right) = \frac{-a}{\lvert u\rvert \sqrt{u^2-a^2}} \cdot \frac{du}{dx}$	**Angle in Q I or Q IV**

Indefinite Integrals

Note: the rules presented in this section omit the " $+C$" term that must be added to all indefinite integrals in order to save space and avoid clutter. Please remember to add the " $+C$" term on all work you perform with indefinite integrals.

Basic Rules

$$\int c\,du = cu$$

$$\int c\,f(u)\,du = c\,f(u)$$

$$\int f(u)+g(u)\,du = \int f(u)\,du + \int g(u)\,du$$

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

Power Rule

$$\int (u^n)\,du = \frac{1}{n+1}\cdot u^{n+1} \quad (n \neq 1)$$

$$\int \frac{1}{u}\,du = \ln|u|$$

Exponential and Logarithmic Functions $(a > 0, a \neq 1)$

$$\int e^u\,du = e^u$$

$$\int \ln u\,dx = u\ln u - u$$

$$\int a^u\,du = \frac{1}{\ln a}\,a^u$$

$$\int \frac{1}{u\ln u}du = \ln(\ln u)$$

Indefinite Integrals of Trigonometric Functions

Trigonometric Functions

$$\int \sin u\, du = -\cos u$$

$$\int \cos u\, du = \sin u$$

$\int \tan u\, du = \ln\lvert\sec u\rvert = -\ln\lvert\cos u\rvert$	$\int \sec^2 u\, du = \tan u$
$\int \cot u\, du = -\ln\lvert\csc u\rvert = \ln\lvert\sin u\rvert$	$\int \csc^2 u\, du = -\cot u$
$\int \sec u\, du = \ln\lvert\sec u + \tan u\rvert$	$\int \sec u \tan u\, du = \sec u$
$\int \csc u\, du = -\ln\lvert\csc u + \cot u\rvert$	$\int \csc u \cot u\, du = -\csc u$

Indefinite Integrals of Inverse Trigonometric Functions

Inverse Trigonometric Functions

$$\int \sin^{-1} u \, du = u \, \sin^{-1} u + \sqrt{1-u^2}$$

$$\int \cos^{-1} u \, du = u \, \cos^{-1} u - \sqrt{1-u^2}$$

$$\int \tan^{-1} u \, du = u \, \tan^{-1} u - \frac{1}{2} \ln(u^2+1)$$

$$\int \cot^{-1} u \, du = u \, \cot^{-1} u + \frac{1}{2} \ln(u^2+1)$$

$$\int \sec^{-1} u \, du = u \, \sec^{-1} u - \ln\left(u + \sqrt{u^2-1}\right) \qquad \sec^{-1} u \in \left\{\left(0, \frac{\pi}{2}\right)\right\}$$

$$= u \, \sec^{-1} u + \ln\left(u + \sqrt{u^2-1}\right) \qquad \sec^{-1} u \in \left\{\left(\frac{\pi}{2}, \pi\right)\right\}$$

$$\int \csc^{-1} u \, du = u \, \csc^{-1} u + \ln\left(u + \sqrt{u^2-1}\right) \qquad \csc^{-1} u \in \left\{\left(0, \frac{\pi}{2}\right)\right\}$$

$$= u \, \csc^{-1} u - \ln\left(u + \sqrt{u^2-1}\right) \qquad \csc^{-1} u \in \left\{\left(-\frac{\pi}{2}, 0\right)\right\}$$

Involving Inverse Trigonometric Functions

$$\int \frac{1}{\sqrt{1-u^2}} \, du = \sin^{-1} u \qquad \int \frac{1}{\sqrt{a^2-u^2}} \, du = \sin^{-1}\left(\frac{u}{a}\right)$$

$$\int \frac{1}{1+u^2} \, du = \tan^{-1} u \qquad \int \frac{1}{a^2+u^2} \, du = \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right)$$

$$\int \frac{1}{u\sqrt{u^2-1}} \, du = \sec^{-1}|u| \qquad \int \frac{1}{u\sqrt{u^2-a^2}} \, du = \frac{1}{a} \sec^{-1}\left(\frac{|u|}{a}\right)$$

Integrals of Special Functions

Selecting the Right Function for an Integral

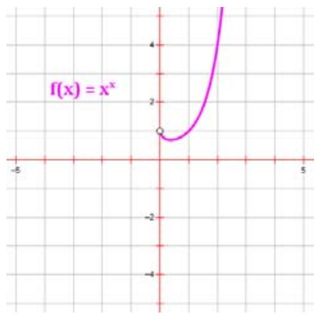
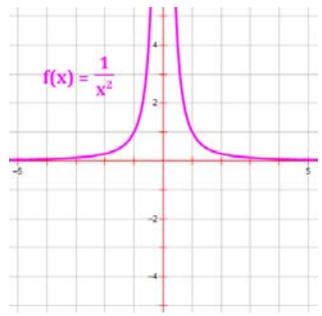
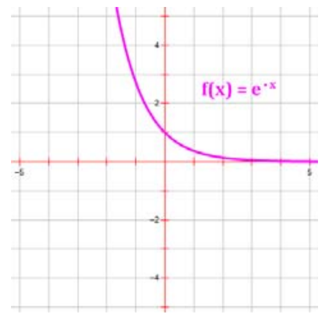
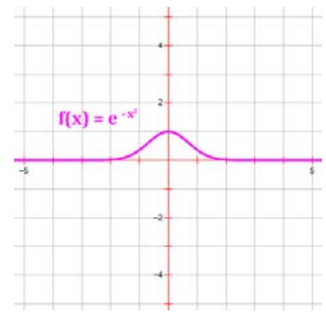
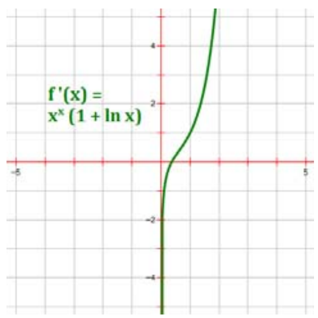
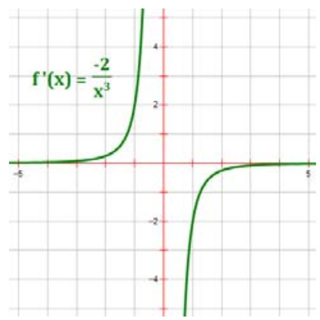
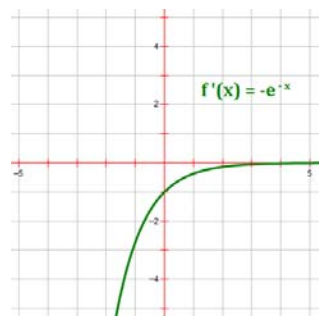
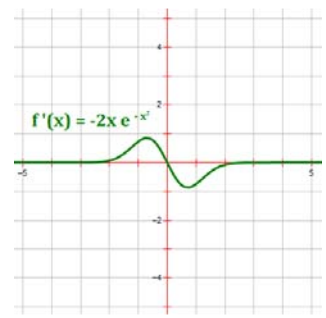
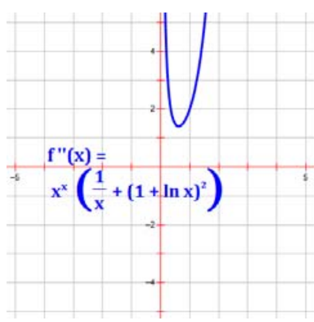
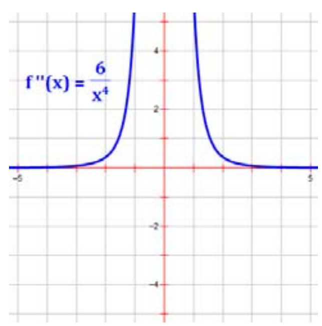
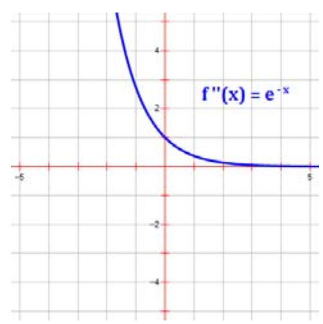
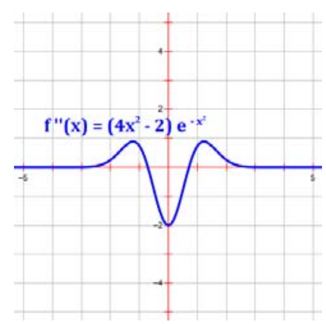
Form	Function	Integral
$\int \frac{1}{\sqrt{a^2-u^2}}\,du$	$\sin^{-1} u$	$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right)$
$\int \frac{1}{a^2+u^2}\,du$	$\tan^{-1} u$	$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)$
$\int \frac{1}{u\sqrt{u^2-a^2}}\,dx$	$\sec^{-1} u$	$\int \frac{1}{u\sqrt{u^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{\lvert u\rvert}{a}\right)$
$\int \frac{1}{\sqrt{u^2+a^2}}\,du$	$\sinh^{-1} u$ *	$\int \frac{1}{\sqrt{u^2+a^2}}\,du = \ln\left(u+\sqrt{u^2+a^2}\right)$
$\int \frac{1}{\sqrt{u^2-a^2}}\,du$	$\cosh^{-1} u$ *	$\int \frac{1}{\sqrt{u^2-a^2}}\,du = \ln\left(u+\sqrt{u^2-a^2}\right)$
$\int \frac{1}{a^2-u^2}\,du \Big\rvert_{a>u}$	$\tanh^{-1} u$ *	$\int \frac{1}{a^2-u^2}\,du = \frac{1}{2a}\ln\left\lvert\frac{a+u}{a-u}\right\rvert$
$\int \frac{1}{u^2-a^2}\,du \Big\rvert_{u>a}$	$\coth^{-1} u$ *	
$\int \frac{1}{u\sqrt{a^2-u^2}}\,du$	$\mathrm{sech}^{-1} u$ *	$\int \frac{1}{u\sqrt{a^2-u^2}}\,du = -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2-u^2}}{\lvert u\rvert}\right)$
$\int \frac{1}{u\sqrt{a^2+u^2}}\,du$	$\mathrm{csch}^{-1} u$ *	$\int \frac{1}{u\sqrt{a^2+u^2}}\,du = -\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2+u^2}}{\lvert u\rvert}\right)$

* This is an inverse hyperbolic function. For more information, see Chapter 6. Note that you do not need to know about inverse hyperbolic functions to use the formulas on this page.

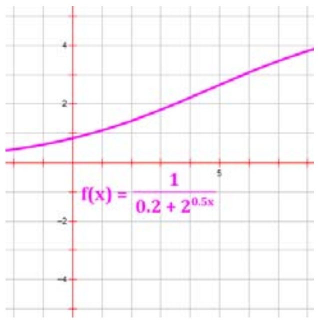
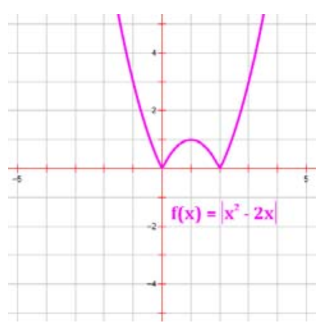
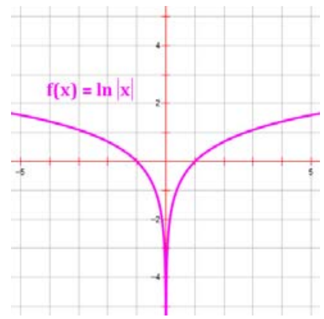
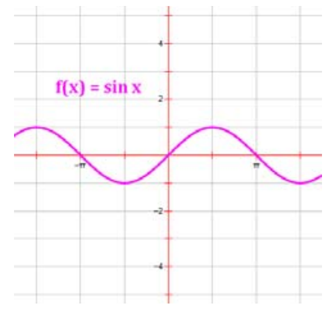
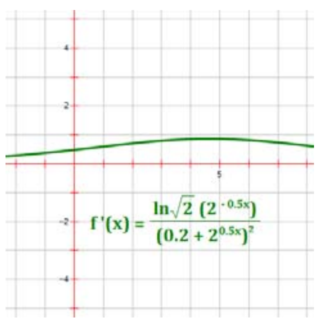
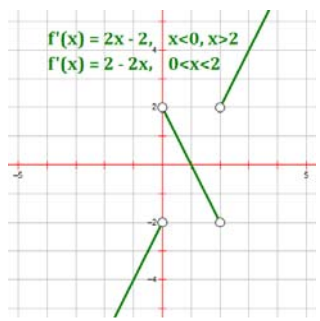
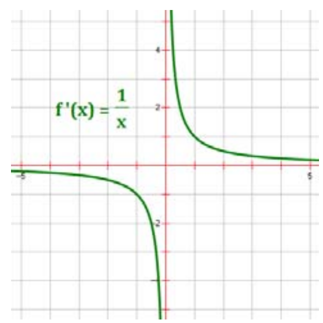
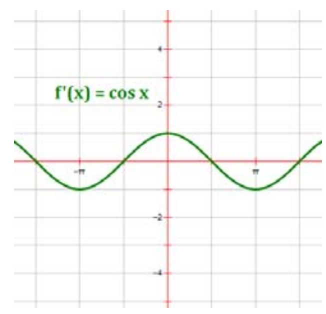
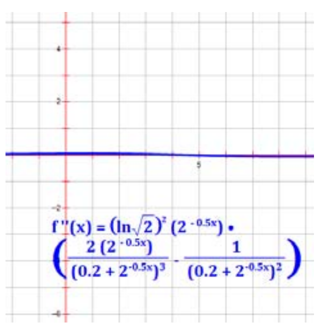
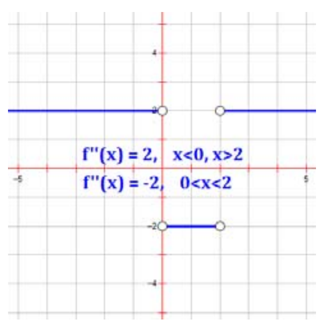
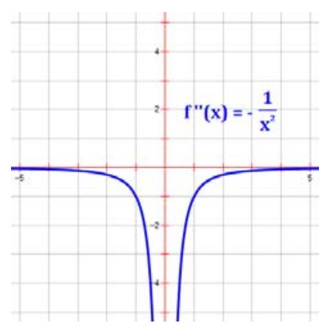
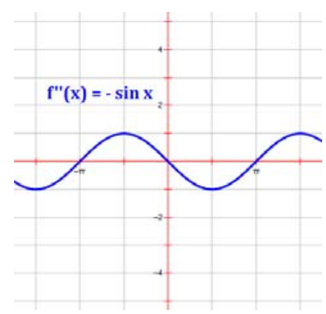
Appendix D

Key Functions and Their Derivatives

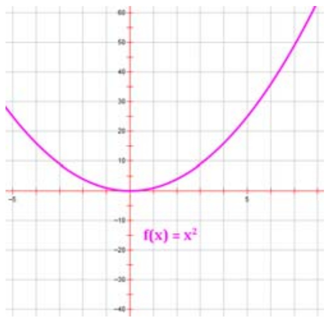
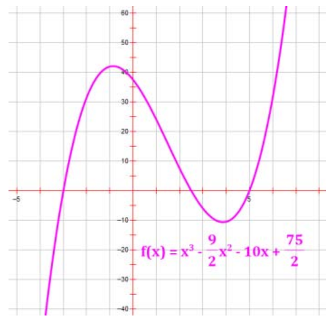
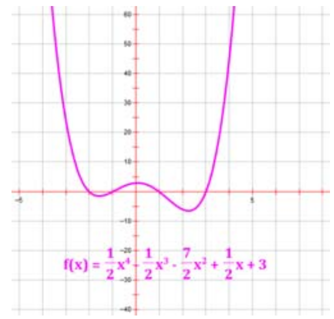
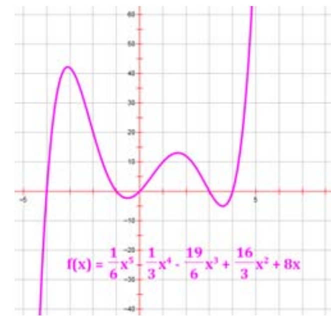
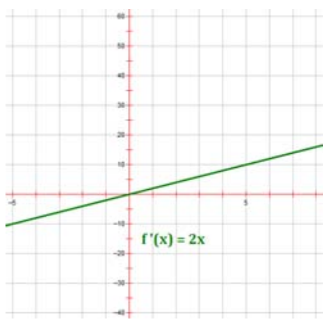
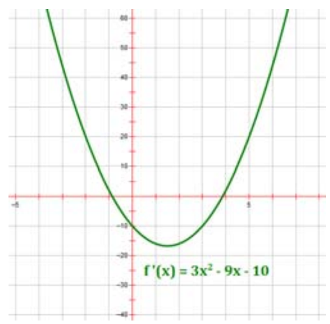
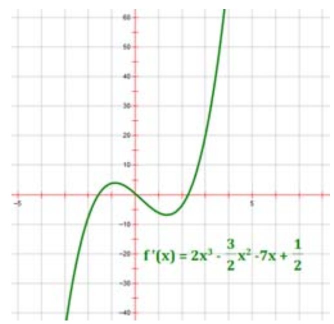
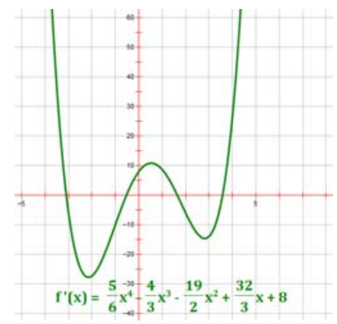
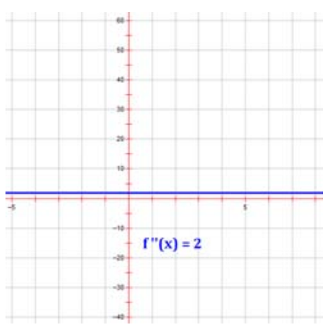
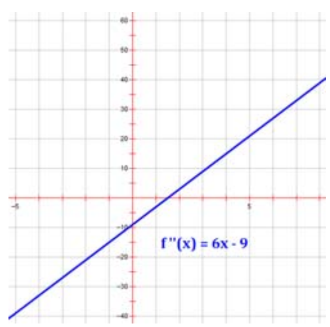
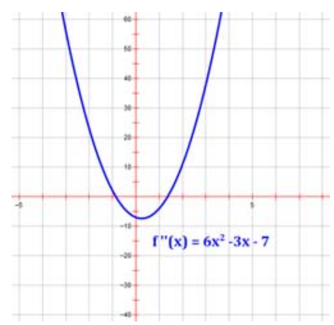
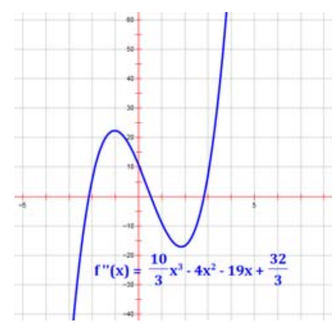
Functions and Their Derivatives

Function	$f(x) = x^x$	$f(x) = \frac{1}{x^2}$	$f(x) = e^{-x}$	$f(x) = e^{-x^2}$
Description	The function is always concave up and the limit of f(x) as x approaches 0 is 1.	The graph of the function has the x- and y-axes as horizontal and vertical asymptotes.	The function is always decreasing and has the x-axis as an asymptote.	The function has one absolute maximum and the x-axis is an asymptote.
Function Graph				
First Derivative Graph				
Second Derivative Graph				

Functions and Their Derivatives

Function	$f(x) = \dfrac{1}{0.2 + 2^{-0.5x}}$	$f(x) = \lvert x^2 - 2x \rvert$	$f(x) = \ln\lvert x \rvert$	$f(x) = \sin x$
Description	The logistic curve. It is always increasing and has one point of inflection.	The function has two relative minima and one relative maximum.	The function is always increasing on the right and always decreasing on the left. The y-axis as an asymptote.	The function is periodic with domain $\mathbb{R}$ and range $[-1, 1]$.
Function Graph				
First Derivative Graph				
Second Derivative Graph				

Functions and Their Derivatives

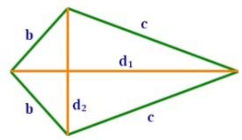
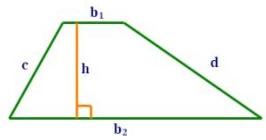
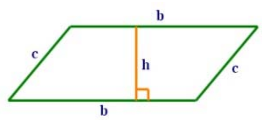
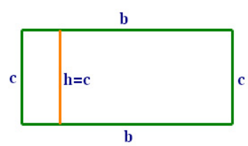
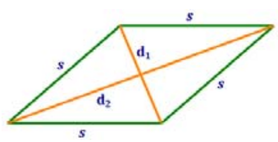
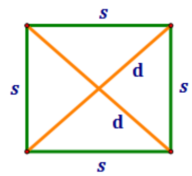
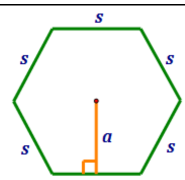
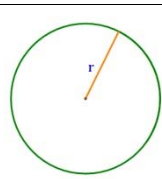
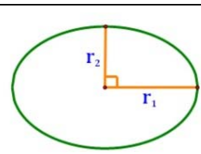
Function	$f(x) = x^2$	$f(x) = \frac{1}{2}(x+3)(2x-5)(x-5)$	$f(x) = \frac{1}{2}(x^2-1)(x+2)(x-3)$	$f(x) = \frac{1}{6}x(x^2-16)(x+1)(x-3)$
Description	The function has one absolute minimum and no points of inflection.	The graph has three zeros, one relative minimum, one relative maximum, and one point of inflection.	The function has one relative maximum, two relative minima, and two points of inflection.	The function has two relative maxima, two relative minima, and three points of inflection.
Function Graph				
First Derivative Graph				
Second Derivative Graph				

Appendix E

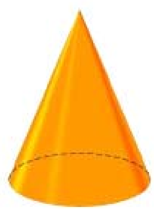
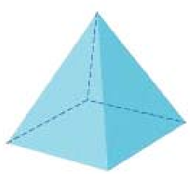
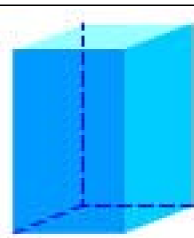
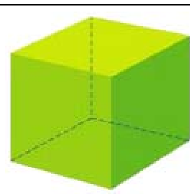
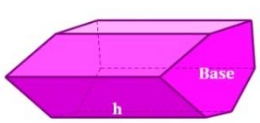
Geometry and Trigonometry Formulas

Geometry

Summary of Perimeter and Area Formulas – 2D Shapes

Shape	Figure	Perimeter	Area
Kite		$P = 2b + 2c$ $b, c = sides$	$A = \frac{1}{2}(d_1 d_2)$ $d_1, d_2 = diagonals$
Trapezoid		$P = b_1 + b_2 + c + d$ $b_1, b_2 = bases$ $c, d = sides$	$\mathrm{A} = \frac{1}{2}(\mathrm{b}_1 + \mathrm{b}_2)\mathrm{h}$ $\mathrm{b}_1, \mathrm{b}_2 = \mathrm{bases}$ $\mathrm{h} = \mathrm{height}$
Parallelogram		$P = 2b + 2c$ $b, c = sides$	$\mathrm{A} = \mathrm{bh}$ $b = base$ $h = height$
Rectangle		$P = 2b + 2c$ $b, c = sides$	$\mathrm{A} = \mathrm{bh}$ $b = base$ $h = height$
Rhombus		$P = 4s$ $s = side$	$A = bh = \frac{1}{2}(d_1 d_2)$ $d_1, d_2 = diagonals$
Square		$P = 4s$ $s = side$	$A = s^2 = \frac{1}{2}(d_1 d_2)$ $d_1, d_2 = diagonals$
Regular Polygon		$P = ns$ $n = number\ of\ sides$ $s = side$	$A = \frac{1}{2}\, a \cdot P$ $a = apothem$ $P = perimeter$
Circle		$C = 2\pi r = \pi d$ $r = radius$ $d = diameter$	$A = \pi r^2$ $r = radius$
Ellipse		$P \approx 2\pi \sqrt{\frac{1}{2}({r_1}^2 + {r_2}^2)}$ $r_1 = major\ axis\ radius$ $r_2 = minor\ axis\ radius$	$A = \pi r_1 r_2$ $r_1 = major\ axis\ radius$ $r_2 = minor\ axis\ radius$

Geometry
Summary of Surface Area and Volume Formulas – 3D Shapes

Shape	Figure	Surface Area	Volume
Sphere		$SA = 4\pi r^2$ $r = radius$	$V = \frac{4}{3}\pi r^3$ $r = radius$
Right Cylinder		$SA = 2\pi rh + 2\pi r^2$ $h = height$ $r = radius\ of\ base$	$V = \pi r^2 h$ $h = height$ $r = radius\ of\ base$
Cone		$SA = \pi rl + \pi r^2$ $l = slant\ height$ $r = radius\ of\ base$	$V = \frac{1}{3}\pi r^2 h$ $h = height$ $r = radius\ of\ base$
Square Pyramid		$SA = 2sl + s^2$ $s = base\ side\ length$ $l = slant\ height$	$V = \frac{1}{3}s^2 h$ $s = base\ side\ length$ $h = height$
Rectangular Prism		$SA = 2 \cdot (lw + lh + wh)$ $l = length$ $w = width$ $h = height$	$V = lwh$ $l = length$ $w = width$ $h = height$
Cube		$SA = 6s^2$ $s = side\ length\ (all\ sides)$	$V = s^3$ $s = side\ length\ (all\ sides)$
General Right Prism		$SA = Ph + 2B$ $P = Perimeter\ of\ Base$ $h = height\ (or\ length)$ $B = area\ of\ Base$	$V = Bh$ $B = area\ of\ Base$ $h = height$

Trigonometry

Function Relationships

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$
$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$
$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Opposite Angle Formulas

$$\sin(-\theta) = -\sin(\theta)$$
$$\cos(-\theta) = \cos(\theta)$$
$$\tan(-\theta) = -\tan(\theta)$$
$$\cot(-\theta) = -\cot(\theta)$$
$$\sec(-\theta) = \sec(\theta)$$
$$\csc(-\theta) = -\csc(\theta)$$

Cofunction Formulas (in Quadrant I)

$$\sin\theta = \cos\left(\frac{\pi}{2}-\theta\right) \qquad \cos\theta = \sin\left(\frac{\pi}{2}-\theta\right)$$
$$\tan\theta = \cot\left(\frac{\pi}{2}-\theta\right) \qquad \cot\theta = \tan\left(\frac{\pi}{2}-\theta\right)$$
$$\sec\theta = \csc\left(\frac{\pi}{2}-\theta\right) \qquad \csc\theta = \sec\left(\frac{\pi}{2}-\theta\right)$$

Angle Addition Formulas

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$
$$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A\tan B}$$
$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$\cot^2\theta + 1 = \csc^2\theta$$

Half Angle Formulas

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$
$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$
$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}$$

Double Angle Formulas

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$
$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

Product-to-Sum Formulas

$$\sin A\cdot\sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$
$$\cos A\cdot\cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$
$$\sin A\cdot\cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$
$$\cos A\cdot\sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

Triple Angle Formulas

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$
$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$
$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$$

Power Reducing Formulas

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$
$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$
$$\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$$

Sum-to-Product Formulas

$$\sin A + \sin B = 2\cdot\sin\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right)$$
$$\sin A - \sin B = 2\cdot\sin\left(\frac{A-B}{2}\right)\cdot\cos\left(\frac{A+B}{2}\right)$$
$$\cos A + \cos B = 2\cdot\cos\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right)$$
$$\cos A - \cos B = -2\cdot\sin\left(\frac{A+B}{2}\right)\cdot\sin\left(\frac{A-B}{2}\right)$$

Arc Length

$S = r\theta$

Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Law of Tangents

$$\frac{a-b}{a+b} = \frac{\tan\left[\frac{1}{2}(A-B)\right]}{\tan\left[\frac{1}{2}(A+B)\right]}$$

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Mollweide's Formulas

$$\frac{a+b}{c} = \frac{\cos\left[\frac{1}{2}(A-B)\right]}{\sin\left(\frac{1}{2}C\right)}$$
$$\frac{a-b}{c} = \frac{\sin\left[\frac{1}{2}(A-B)\right]}{\cos\left(\frac{1}{2}C\right)}$$

Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta = \text{cis}\,\theta$$

DeMoivre's Formula

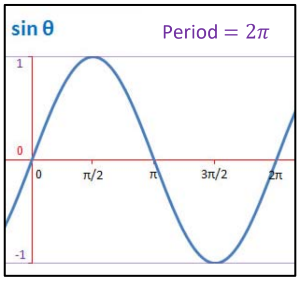
$$(r\ \text{cis}\,\theta)^n = r^n\ \text{cis}\,(n\theta)$$

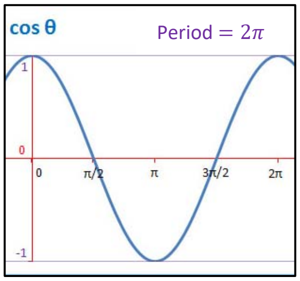
Polar Multiplication and Division

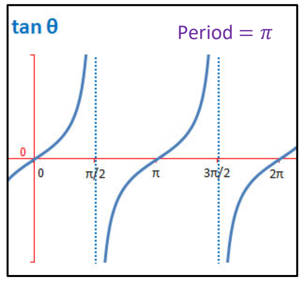
Let: $a = r_1\ \text{cis}\,\theta \qquad b = r_2\ \text{cis}\,\varphi$

$$a\cdot b = r_1 r_2\ \text{cis}\,(\theta+\varphi) \qquad \frac{a}{b} = \frac{r_1}{r_2}\ \text{cis}\,(\theta-\varphi)$$

Trigonometry







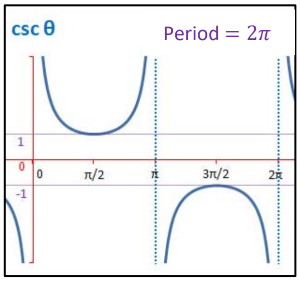
$\boldsymbol{y = A \cdot f(Bx - C) + D}$

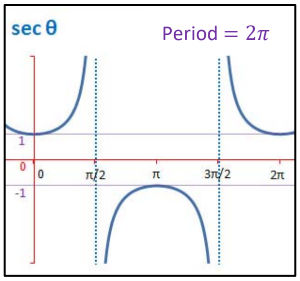
Amplitude: $|A|$

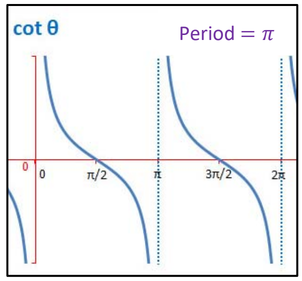
Period: $\dfrac{parent\ "f"\ period}{B}$

Phase Shift: $\dfrac{C}{B} \longrightarrow$

Vertical Shift: D







Harmonic Motion

$d = a \cos \omega t$ or

$d = a \sin \omega t$

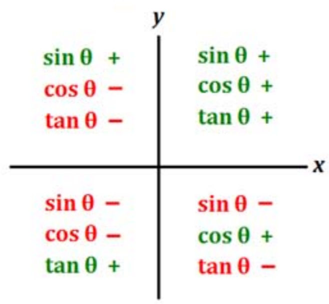
$f = \dfrac{1}{\text{period}} = \dfrac{\omega}{2\pi}$

$\omega = 2\pi f, \quad \omega > 0$

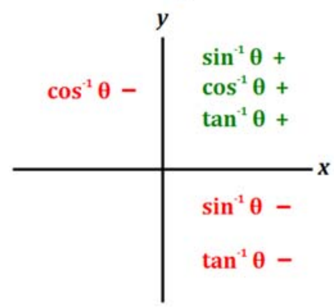
Trig Functions of Special Angles (Unit Circle)

θ Rad	$\theta°$	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0^o	0	1	0
$\pi/6$	30^o	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
$\pi/4$	45^o	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$\pi/3$	60^o	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$\pi/2$	90^o	1	0	*undefined*

Signs of Trig Functions by Quadrant



Locations of Principal Values of Inverse Trig Functions



Rectangular/Polar Conversion

Rectangular	Polar
(x, y)	(r, θ)
$x = r\cos\theta$ $y = r\sin\theta$	$r = \sqrt{x^2 + y^2}$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$
$a + bi$	$r\,(\cos\theta + i\sin\theta)$ or $r\ cis\theta$
$a = r\cos\theta$ $b = r\sin\theta$	$r = \sqrt{a^2 + b^2}$ $\theta = \tan^{-1}\left(\frac{b}{a}\right)$
$a\mathbf{i} + b\mathbf{j}$	$\lVert\mathbf{v}\rVert \angle\theta$
$a = \lVert\mathbf{v}\rVert\cos\theta$ $b = \lVert\mathbf{v}\rVert\sin\theta$	$\lVert\mathbf{v}\rVert = \sqrt{a^2 + b^2}$ $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

Triangle Area

$$A = \frac{1}{2}bh$$

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

$$s = \frac{1}{2}P = \frac{1}{2}(a + b + c)$$

$$A = \frac{1}{2}\left(\frac{a^2 \sin B \sin C}{\sin A}\right)$$

$$A = \frac{1}{2}\,ab\sin C$$

$$A = \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$$

$$A = \frac{1}{2}\ \lVert\mathbf{u}\rVert\ \lVert\mathbf{v}\rVert\ \sin\theta$$

Vector Properties

$0 + \mathbf{u} = \mathbf{u} + 0 = \mathbf{u}$

$\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = 0$

$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

$m(n\mathbf{u}) = (mn)\mathbf{u}$

$m(\mathbf{u} + \mathbf{v}) = m\mathbf{u} + m\mathbf{v}$

$(m + n)\mathbf{u} = m\mathbf{u} + n\mathbf{u}$

$1(\mathbf{v}) = \mathbf{v}$

$\lVert m\mathbf{v}\rVert = |m|\ \lVert\mathbf{v}\rVert$

Unit Vector: $\dfrac{\mathbf{v}}{\lVert\mathbf{v}\rVert}$

Vector Dot Product

$\mathbf{u} \circ \mathbf{v} = (u_1 \cdot v_1) + (u_2 \cdot v_2)$

$\mathbf{u} \circ (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \circ \mathbf{v}) + (\mathbf{u} \circ \mathbf{w})$

Vector Cross Product

$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = u_1 v_2 - u_2 v_1$

$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

Angle between Vectors

$\cos\theta = \dfrac{\mathbf{u} \circ \mathbf{v}}{\lVert\mathbf{u}\rVert\ \lVert\mathbf{v}\rVert}$ $\quad \sin\theta = \dfrac{\lVert\mathbf{u} \times \mathbf{v}\rVert}{\lVert\mathbf{u}\rVert\ \lVert\mathbf{v}\rVert}$

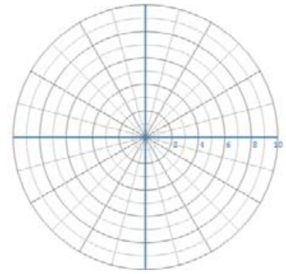
$\perp$ iff $\mathbf{u} \circ \mathbf{v} = 0$ $\quad \parallel$ iff $\mathbf{u} \times \mathbf{v} = 0$

Appendix F

Polar and Parametric Equations

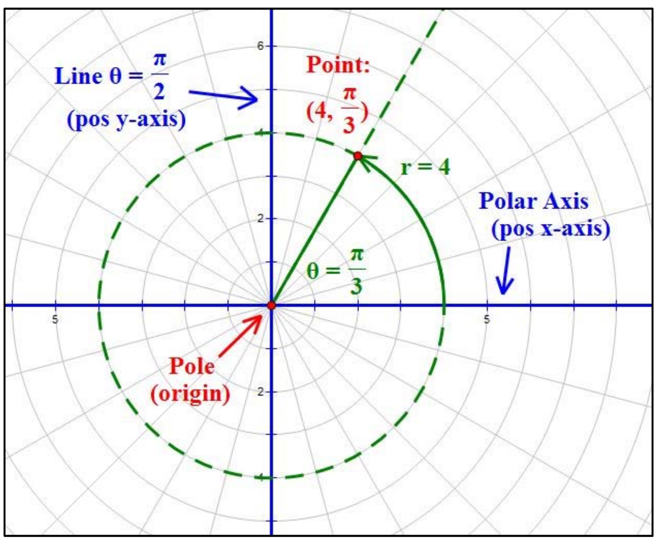
Polar Graphs

Typically, Polar Graphs will be plotted on polar graph paper such as that illustrated at right. On this graph, a point (r, θ) can be considered to be the intersection of the circle of radius r and the terminal side of the angle θ (see the illustration below). Note: a free PC app that can be used to design and print your own polar graph paper is available at www.mathguy.us.



Parts of the Polar Graph

The illustration below shows the key parts of a polar graph, along with a point, $\left(4, \frac{\pi}{3}\right)$.



The Pole is the point $(0, 0)$ (i.e., the origin).

The Polar Axis is the positive x-axis.

The Line: $\theta = \frac{\pi}{2}$ is the positive y-axis.

Many equations that contain the cosine function are symmetric about the x-axis.

Many equations that contain the sine function are symmetric about the y-axis.

Polar Equations – Symmetry

Following are the three main types of symmetry exhibited in many polar equation graphs:

Symmetry about:	Quadrants Containing Symmetry	Symmetry Test[1]
Pole	Opposite (I and III or II and IV)	Replace r with $-r$ in the equation
x-axis	Left hemisphere (II and III) or right hemisphere (I and IV)	Replace θ with $-\theta$ in the equation
y-axis	Upper hemisphere (I and II) or lower hemisphere (III and IV)	Replace (r, θ) with $(-r, -\theta)$ in the equation

(1) If performing the indicated replacement results in an equivalent equation, the equation passes the symmetry test and the indicated symmetry exists. If the equation fails the symmetry test, symmetry may or may not exist.

Graphs of Polar Equations

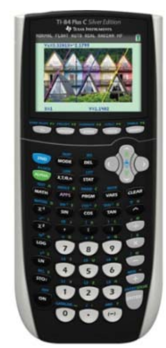
Graphing Methods

Method 1: Point plotting

- Create a two-column chart that calculates values of r for selected values of θ. This is akin to a two-column chart that calculates values of y for selected values of x that can be used to plot a rectangular coordinates equation (e.g., $y = x^2 - 4x + 3$).
- The θ-values you select for purposes of point plotting should vary depending on the equation you are working with (in particular, the coefficient of θ in the equation). However, a safe bet is to start with multiples of $\pi/_6$ (including $\theta = 0$). Plot each point on the polar graph and see what shape emerges. If you need more or fewer points to see what curve is emerging, adjust as you go.
- If you know anything about the curve (typical shape, symmetry, etc.), use it to facilitate plotting points.
- Connect the points with a smooth curve. Admire the result; many of these curves are aesthetically pleasing.

Method 2: Calculator

Using a TI-84 Plus Calculator or its equivalent, do the following:

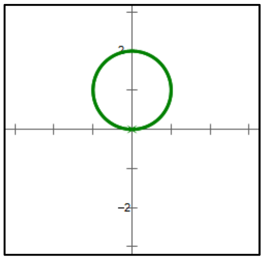


- Make sure your calculator is set to radians and polar functions. Hit the **MODE** key; select **RADIANS** in row 4 and **POLAR** in row 5. After you do this, hitting **CLEAR** will get you back to the main screen.
- Hit **Y=** and enter the equation in the form $r = f(\theta)$. Use the **X,T,θ,n** key to enter θ into the equation. If your equation is of the form $r^2 = f(\theta)$, you may need to enter two functions, $r = \sqrt{f(\theta)}$ and $r = -\sqrt{f(\theta)}$, and plot both.
- Hit **GRAPH** to plot the function or functions you entered in the previous step.
- If necessary, hit **WINDOW** to adjust the parameters of the plot.
  - If you cannot see the whole function, adjust the **X-** and **Y-** variables (or use **ZOOM**).
  - If the curve is not smooth, reduce the value of the **θstep** variable. This will plot more points on the screen. Note that smaller values of **θstep** require more time to plot the curve, so choose a value that plots the curve well in a reasonable amount of time.
  - If the entire curve is not plotted, adjust the values of the **θmin** and **θmax** variables until you see what appears to be the entire plot.

Note: You can view the table of points used to graph the polar function by hitting **2ND – TABLE**.

Graph of Polar Equations

Circle

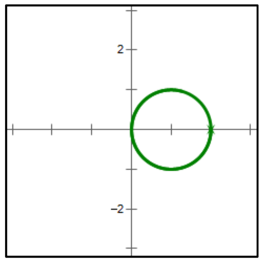


Equation: $r = a\sin\theta$

Location:
above x-axis if $a > 0$
below x-axis if $a < 0$

Radius: $a/2$

Symmetry: y-axis

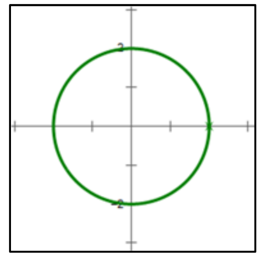


Equation: $r = a\cos\theta$

Location:
right of y-axis if $a > 0$
left of y-axis if $a < 0$

Radius: $a/2$

Symmetry: x-axis



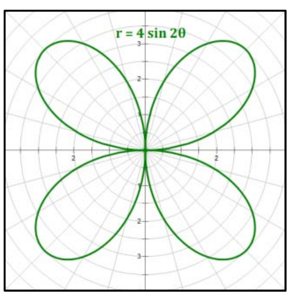
Equation: $r = a$

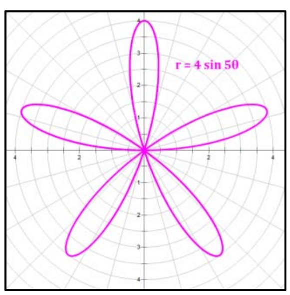
Location:
Centered on the Pole

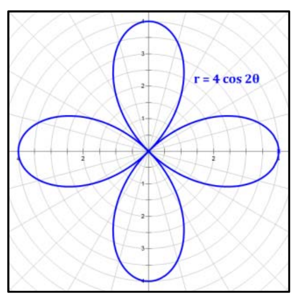
Radius: a

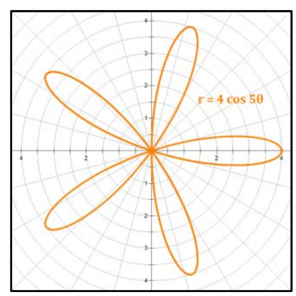
Symmetry: Pole, x-axis, y-axis

Rose







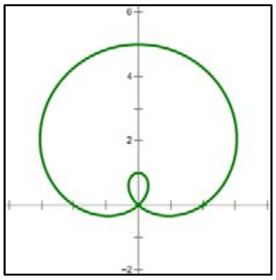


Characteristics of roses:

- Equation: $r = a\sin n\theta$
  - Symmetric about the y-axis
- Equation: $r = a\cos n\theta$
  - Symmetric about the x-axis
- Contained within a circle of radius $r = a$
- If n is odd, the rose has n petals.
- If n is even the rose has $2n$ petals.
- Note that a circle is a rose with one petal (i.e, $n = 1$).

Graphs of Polar Equations

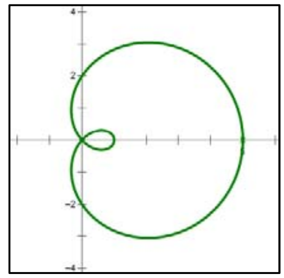
Limaçon of Pascal



Equation: $r = a + b\sin\theta$

Location: bulb above x-axis if $b > 0$
bulb below x-axis if $b < 0$

Symmetry: y-axis

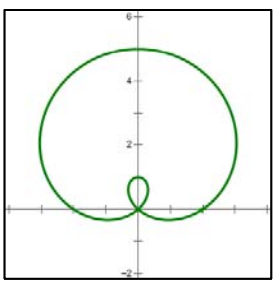


Equation: $r = a + b\cos\theta$

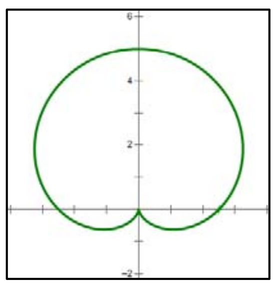
Location: bulb right of y-axis if $b > 0$
bulb left of y-axis if $b < 0$

Symmetry: x-axis

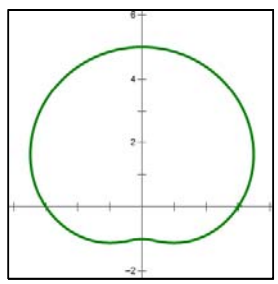
Four Limaçon Shapes



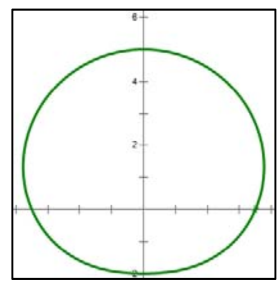
$a < b$
Inner loop



$a = b$
"Cardioid"

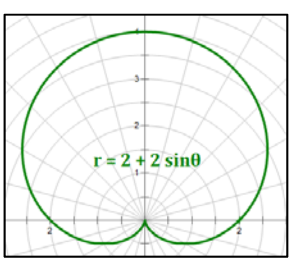


$b < a < 2b$
Dimple

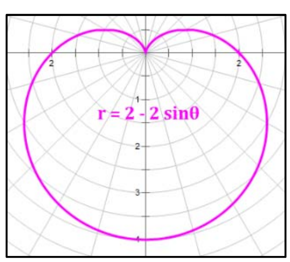


$a \geq 2b$
No dimple

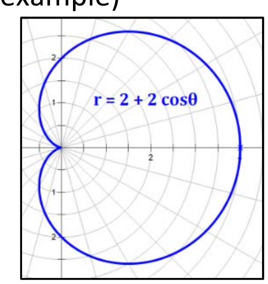
Four Limaçon Orientations (using the Cardioid as an example)



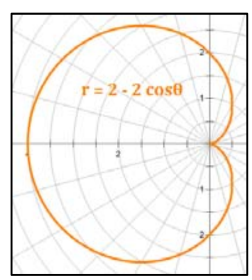
sine function
$b > 0$



sine function
$b < 0$



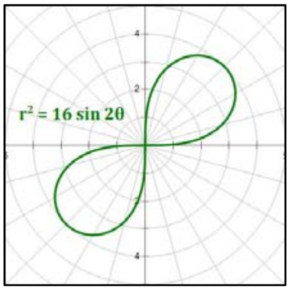
cosine function
$b > 0$

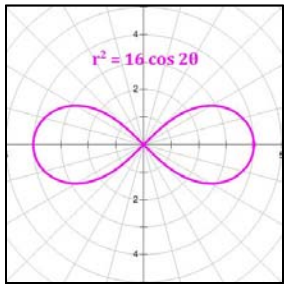


cosine function
$b < 0$

Graph of Polar Equations

Lemniscate of Bernoulli



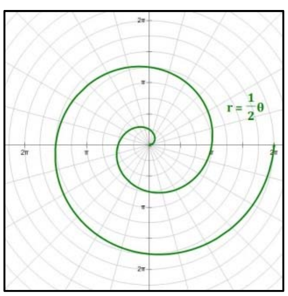


The lemniscate is the set of all points for which the product of the distances from two points (i.e., foci) which are "2c" apart is c^2.

Characteristics of lemniscates:

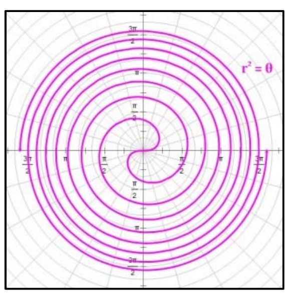
- Equation: $r^2 = a^2 \sin 2\theta$
  - Symmetric about the line $y = x$
- Equation: $r^2 = a^2 \cos 2\theta$
  - Symmetric about the x-axis
- Contained within a circle of radius $r = a$

Spirals



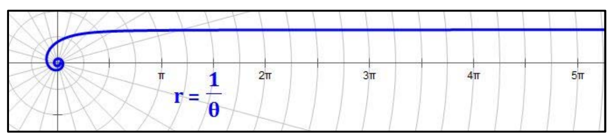
Archimedes' Spiral

$r = a\theta$

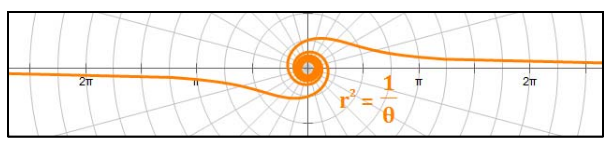


Fermat's Spiral

$r^2 = a^2\theta$



Hyperbolic Spiral: $r = \dfrac{a}{\theta}$



Lituus: $r^2 = \dfrac{a^2}{\theta}$

Characteristics of spirals:

- Equation: $r^b = a^b\theta,\ b > 0$
  - Distance from the Pole increases with θ
- Equation: $r^b = \dfrac{a^b}{\theta},\ b > 0$
  - Hyperbolic Spiral ($b = 1$): asymptotic to the line a units from the x-axis
  - Lituus ($b = 2$): asymptotic to the x-axis
- Not contained within any circle

Graphing Polar Equations – The Rose

Example F.1: $\mathbf{r = 4\sin 2\theta}$

This function is a rose. Consider the forms $r = a\sin b\theta$ and $r = a\cos b\theta$.

The number of petals on the rose depends on the value of b.

- If b is an even integer, the rose will have $2b$ petals.
- If b is an odd integer, it will have b petals.

Let's create a table of values and graph the equation:

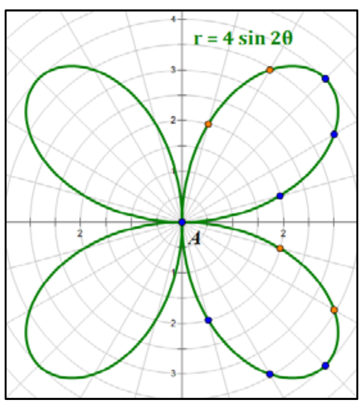
$\mathbf{r = 4\sin 2\theta}$			
$\boldsymbol{\theta}$	$\boldsymbol{r}$	$\boldsymbol{\theta}$	$\boldsymbol{r}$
0	0		
$\pi/12$	2	$7\pi/12$	-2
$\pi/6$	3.464	$2\pi/3$	-3.464
$\pi/4$	4	$3\pi/4$	-4
$\pi/3$	3.464	$5\pi/6$	-3.464
$5\pi/12$	2	$11\pi/12$	-2
$\pi/2$	0	π	0

Because this function involves an argument of 2θ, we want to start by looking at values of θ in $[0, 2\pi] \div 2 = [0, \pi]$. You could plot more points, but this interval is sufficient to establish the nature of the curve; so you can graph the rest easily.

Once symmetry is established, these values are easily determined.

The values in the table generate the points in the two petals right of the y-axis.

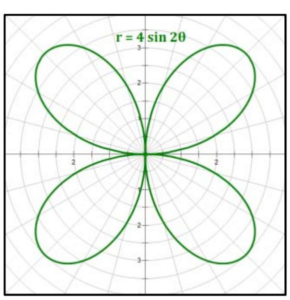
Knowing that the curve is a rose allows us to graph the other two petals without calculating more points.

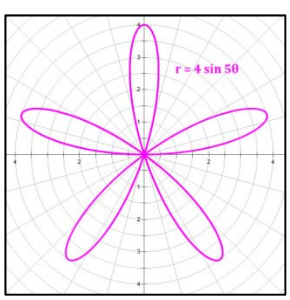


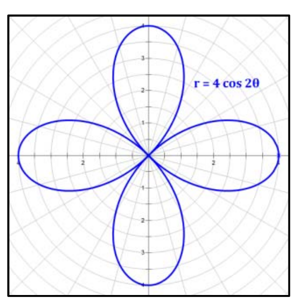
Blue points on the graph correspond to blue values in the table.

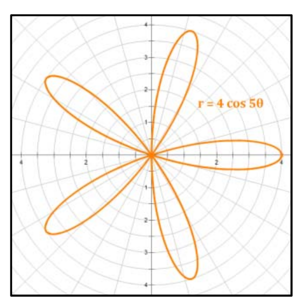
Orange points on the graph correspond to orange values in the table.

The four Rose forms:









Graphing Polar Equations – The Cardioid

Example F.2: $\mathbf{r = 2 + 2\sin\theta}$

This cardioid is also a limaçon of form $r = a + b\sin\theta$ with $a = b$. The use of the sine function indicates that the large loop will be symmetric about the y-axis. The $+$ sign indicates that the large loop will be above the x-axis. Let's create a table of values and graph the equation:

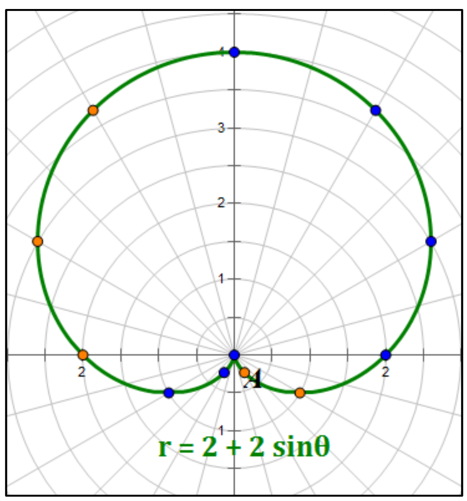
$\mathbf{r = 2 + 2\sin\theta}$			
θ	r	θ	r
0	2		
$\pi/6$	3	$7\pi/6$	1
$\pi/3$	3.732	$4\pi/3$	0.268
$\pi/2$	4	$3\pi/2$	0
$2\pi/3$	3.732	$5\pi/3$	0.268
$5\pi/6$	3	$11\pi/6$	1
π	2	2π	2

Generally, you want to look at values of θ in $[0, 2\pi]$. However, some functions require larger intervals. The size of the interval depends largely on the nature of the function and the coefficient of θ.

Once symmetry is established, these values are easily determined.

The portion of the graph above the x-axis results from θ in Q1 and Q2, where the sine function is positive.

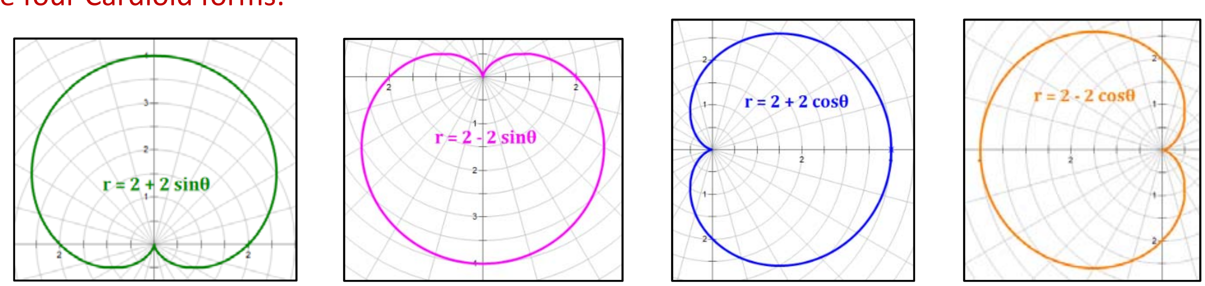
Similarly, the portion of the graph below the x-axis results from θ in Q3 and Q4, where the sine function is negative.



Blue points on the graph correspond to blue values in the table.

Orange points on the graph correspond to orange values in the table.

The four Cardioid forms:



Converting Between Polar and Rectangular Forms of Equations

Rectangular to Polar

To convert an equation from Rectangular Form to Polar Form, use the following equivalences:

$x = r\cos\theta$ Substitute $r\cos\theta$ for x

$y = r\sin\theta$ Substitute $r\sin\theta$ for y

$x^2 + y^2 = r^2$ Substitute r^2 for $x^2 + y^2$

Example F.3: Convert $8x - 3y + 10 = 0$ to a polar equation of the form $r = f(\theta)$.

Starting Equation: $8x - 3y + 10 = 0$

Substitute $x = r\cos\theta$ and $y = r\sin\theta$: $8 \cdot r\cos\theta - 3 \cdot r\sin\theta + 10 = 0$

Factor out r: $r\,(8\cos\theta - 3\sin\theta) = -10$

Divide by $(8\cos\theta - 3\sin\theta)$: $\boldsymbol{r = \dfrac{-10}{8\cos\theta - 3\sin\theta}}$

Polar to Rectangular

To convert an equation from Polar Form to Rectangular Form, use the following equivalences:

$\cos\theta = \dfrac{x}{r}$ Substitute $\dfrac{x}{r}$ for $\cos\theta$

$\sin\theta = \dfrac{y}{r}$ Substitute $\dfrac{y}{r}$ for $\sin\theta$

$r^2 = x^2 + y^2$ Substitute $x^2 + y^2$ for r^2

Example F.4: Convert r = 8 cos θ + 9 sin θ to a rectangular equation.

Starting Equation: r = 8 cos θ + 9 sin θ

Substitute $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$: $r = 8\left(\frac{x}{r}\right) + 9\left(\frac{y}{r}\right)$

Multiply by r: $r^2 = 8x + 9y$

Substitute $r^2 = x^2 + y^2$: $x^2 + y^2 = 8x + 9y$

Subtract $8x + 9y$: $x^2 - 8x \quad + y^2 - 9y \quad = 0$

Complete the square: $(x^2 - 8x + 16) + \left(y^2 - 9y + \frac{81}{4}\right) = 16 + \frac{81}{4}$

Simplify to standard form for a circle: $\boldsymbol{(x-4)^2 + \left(y - \frac{9}{2}\right)^2 = \frac{145}{4}}$

Parametric Equations

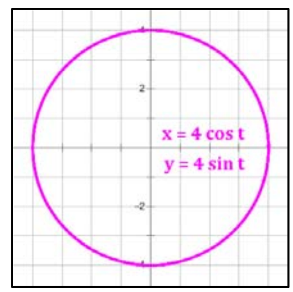
One way to define a curve is by making x and y (or r and θ) functions of a third variable, often t (for time). The third variable is called the Parameter, and functions defined in this manner are said to be in Parametric Form. The equations that define the desired function are called Parametric Equations.

In Parametric Equations, the parameter is the independent variable. Each of the other two (or more) variables is dependent on the value of the parameter. As the parameter changes, the other variables change, generating the points of the function.

Example F.5: A relatively simple example is a circle, which we can define as follows:

Circle: $x = r\cos t \quad y = r\sin t$

As the variable t progresses from 0 to 2π, a circle of radius r is born.



The circle in the illustration at right can be defined in several ways:

Cartesian form: $x^2 + y^2 = 16$

Polar form: $r = 4$

Parametric form: $x = 4\cos t \qquad y = 4\sin t$

Familiar Curves

Many curves with which the student may be familiar have parametric forms. Among those are the following:

Curve	Cartesian Form	Polar Form	Parametric Form
Parabola with horizontal directrix	$y = a(x-h)^2 + k$	$r = \dfrac{p}{1 \pm \sin\theta}$	$x = 2pt$ $y = pt^2$
Ellipse with horizontal major axis	$\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$	$r = \dfrac{ep}{1 \pm e\cdot\cos\theta}$ $(0 < e < 1)$	$x = a\cos t$ $y = b\sin t$
Hyperbola with horizontal transverse axis	$\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$	$r = \dfrac{ep}{1 \pm e\cdot\cos\theta}$ $(e > 1)$	$x = a\sec t$ $y = b\tan t$

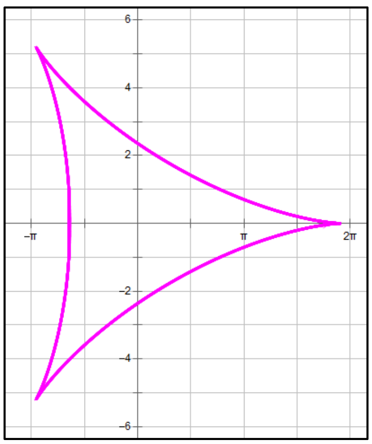
As can be seen from this chart, sometimes the parametric form of a function is its simplest. In fact, parametric equations often allow us to graph curves that would be very difficult to graph in either Polar form or Cartesian form. Some of these are illustrated on the next page.

Some Functions Defined by Parametric Equations

(Star Wars fans: are these the "oids" you are looking for?)

The graphs below are examples of functions defined by parametric equations. The equations and a brief description of the curve are provided for each function.

Deltoid

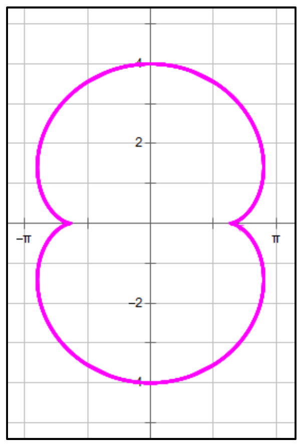


Parametric equations:

$$x = 2a\cos t + a\cos 2t$$
$$y = 2a\sin t - a\sin 2t$$

The deltoid is the path of a point on the circumference of a circle as it makes three complete revolutions on the inside of a larger circle.

Nephroid

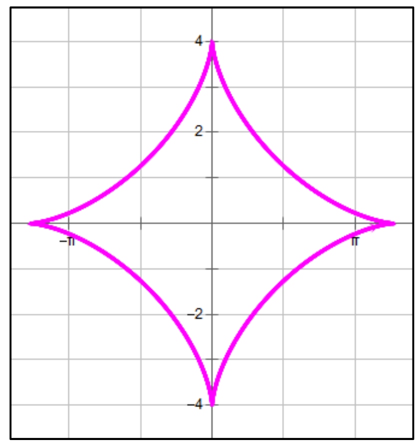


Parametric equations:

$$x = a(3\cos t - \cos 3t)$$
$$y = a(3\sin t - \sin 3t)$$

The nephroid is the path of a point on the circumference of a circle as it makes two complete revolutions on the outside of a larger circle.

Astroid

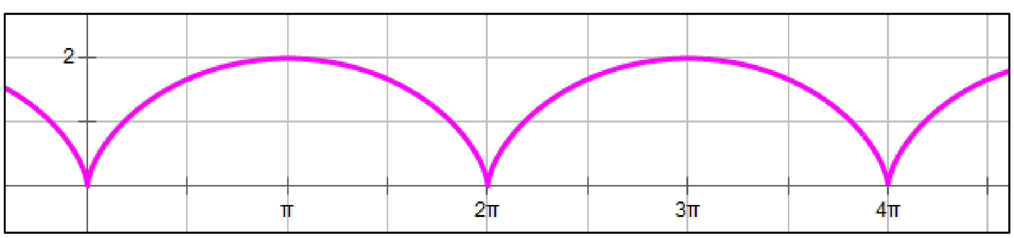


Parametric equations:

$$x = a\cos^3 t$$
$$y = a\sin^3 t$$

The astroid is the path of a point on the circumference of a circle as it makes four complete revolutions on the inside of a larger circle.

Cycloid



Parametric equations:

$$x = a(t - \sin t)$$
$$y = a\,(1 - \cos t)$$

The cycloid is the path of a point on the circumference of a circle as the circle rolls along a flat surface (think: the path of a point on the outside of a bicycle tire as you ride on the sidewalk). The cycloid is both a *brachistochrone* and a *tautochrone* (look these up if you are interested).

Appendix G
Interesting Series and Summation Formulas

$\sum_{k=1}^{n}(c) = nc$	$c + c + \cdots + c = nc$
$\sum_{k=1}^{n}(k) = \frac{n(n+1)}{2}$	$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$
$\sum_{k=1}^{n}(k^2) = \frac{n(n+1)(2n+1)}{6}$	$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n}(k^3) = \left(\frac{n(n+1)}{2}\right)^2$	$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$
$\sum_{k=0}^{\infty}(x^k) = \frac{1}{1-x} \quad for\ \{-1 < x < 1\}$	$1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x}$
$\sum_{k=0}^{\infty}\left(\frac{x^k}{k!}\right) = e^x$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = e^x$
$\sum_{k=1}^{\infty}\left[\frac{1}{k}\left(\frac{x-1}{x}\right)^k\right] = \ln x \quad for\ x \geq \frac{1}{2}$	$\left(\frac{x-1}{x}\right) + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \cdots = \ln x$
$\sum_{k=1}^{\infty}\left[(-1)^{(k+1)}\left(\frac{x^k}{k}\right)\right] = \ln(1+x)$ $for\ \{-1 < x \leq 1\}$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \ln(1+x)$
$\sum_{k=0}^{\infty}\left[(-1)^k\left(\frac{x^{(2k)}}{(2k)!}\right)\right] = \cos x$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \cos x$
$\sum_{k=0}^{\infty}\left[(-1)^k\left(\frac{x^{(2k+1)}}{(2k+1)!}\right)\right] = \sin x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sin x$
$\sum_{k=0}^{\infty}\left[(-1)^k\left(\frac{x^{(2k+1)}}{(2k+1)}\right)\right] = \tan^{-1} x$ $for\ \{-1 \leq x \leq 1\}$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \tan^{-1} x$

Calculus Handbook
Index

Calculus Handbook
Index

Calculus Handbook
Index

Calculus Handbook
Index

Calculus Handbook

Index

Calculus Handbook
Index

Calculus Handbook
Index

Calculus Handbook
Index